EXAM III   MATH 232, CALCULUS III, FALL 2016   NAME: SOLUTION

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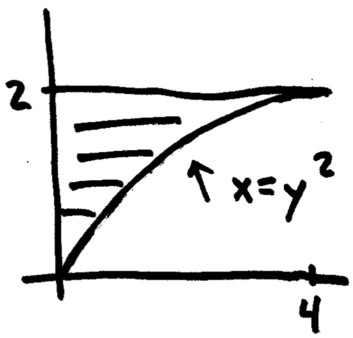
1. Evaluate $\int_0^4 \int_{\sqrt{x}}^2 \cos(y^3)\, dy\, dx$. Hint: Reverse the order of integration.



$0 \le y \le 2$
$0 \le x \le y^2$

$$\int_0^2 \int_0^{y^2} \cos(y^3)\, dx\, dy$$

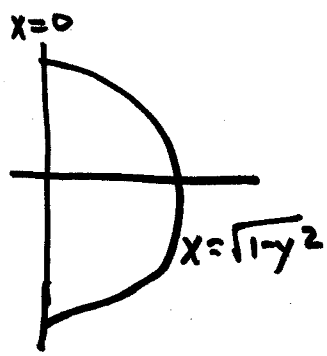
INSIDE: $\displaystyle\int_0^{y^2} \cos(y^3)\, dx = x\cos(y^3)\Big|_0^{y^2} = y^2\cos(y^3)$

OUTSIDE: $\displaystyle\int_0^2 y^2\cos(y^3)\, dy$ LET $u = y^3$, $\frac{1}{3}du = y^2 dy$

$$= \int_0^8 \frac{1}{3}\cos(u)\, du = \frac{1}{3}\sin(u)\Big|_{u=0}^{u=8}$$

$$= \frac{1}{3}\sin 8$$

2. Set up $\displaystyle\int\int_D \frac{x^2}{x^2+y^2}\, dA$, where D is the region bounded by $x = 0$ and $x = \sqrt{1-y^2}$, by changing to polar coordinates.



$0 \le r \le 1$
$-\frac{\pi}{2} \le \theta \le \pi/2$

$x^2 = r^2\cos^2\theta$
$x^2+y^2 = r^2$

$$\int_0^1 \int_{-\pi/2}^{\pi/2} \frac{r^2\cos^2\theta}{r^2}\, r\, d\theta\, dr$$

$$= \int_0^1 \int_{-\pi/2}^{\pi/2} r\cos^2\theta\, d\theta\, dr$$

3. Evaluate $\int_{-2}^{4}\int_{0}^{8}\int_{1}^{e^x} \frac{y}{z}\, dz\, dy\, dx$. Make sure you simplify as you integrate!

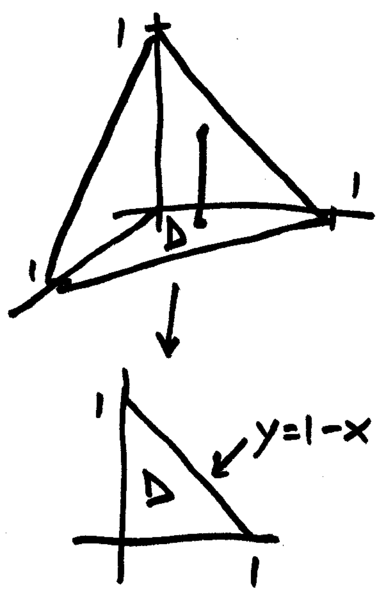
INSIDE: $\int_1^{e^x} \frac{y}{z}\,dz = y\ln|z|\Big|_{z=1}^{z=e^x} = y\ln|e^x| - y\ln 1 = yx$

MIDDLE: $\int_0^8 yx\,dy = \frac{1}{2}y^2x\Big|_{y=0}^{y=8} = 32x$

OUTSIDE: $\int_{-2}^{4} 32x\,dx = 16x^2\Big|_{-2}^{4} = 16(4)^2 - 16(-2)^2$

$= 192$

4. Set up but do not evaluate the volume of the solid tetrahedron which is bounded by the first octant and the plane $x + y + z = 1$ using a triple integral.



$0 \le x \le 1$

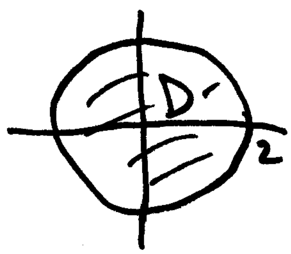
$0 \le y \le 1-x$

$0 \le z \le 1-x-y$

$V = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1\, dz\,dy\,dx$

5. Find the volume of the solid which is bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3 - y$.

$$\iint_D \int_0^{3-y} 1\, dz\, dA = \int_0^2 \int_0^{2\pi} \int_0^{3-r\sin\theta} 1 r\, dz\, d\theta\, dr$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$0 \le z \le 3 - r\sin\theta$

INSIDE: $\displaystyle\int_0^{3-r\sin\theta} r\, dz = rz\Big|_{z=0}^{z=3-r\sin\theta} = r(3-r\sin\theta)$

$= 3r - r^2\sin\theta$

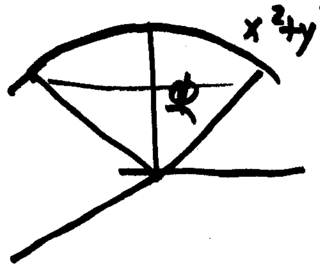
MIDDLE: $\displaystyle\int_0^{2\pi} 3r - r^2\sin\theta\, d\theta = 3r\theta + r^2\cos\theta\Big|_0^{2\pi}$

$= [3r(2\pi) + \cancel{r^2\cos(2\pi)}] - [0 + \cancel{r^2\cos 0}]$

$= 6\pi r$

OUTSIDE: $\displaystyle\int_0^2 6\pi r\, dr = 3\pi r^2\Big|_0^2 = 12\pi$

6. Evaluate $\displaystyle\iiint_E z\sqrt{x^2+y^2+z^2}\, dV$ using spherical coordinates where E is the region below the sphere $x^2+y^2+z^2 = 1$ and above the cone $\phi = \pi/6$.



$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/6$

$z\sqrt{x^2+y^2+z^2} = \rho\cos\phi \cdot \sqrt{\rho^2}$

$= \rho^2\cos\phi$

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^2\cos\phi \cdot \rho^2\sin\phi\, d\phi\, d\theta\, d\rho$$

- $\displaystyle\int_0^1 \rho^4 d\rho = \frac{1}{5}\rho^5\Big|_0^1 = \frac{1}{5}$

- $\displaystyle\int_0^{\pi/6} \cos\phi\sin\phi\, d\phi = \frac{1}{2}\sin^2\phi\Big|_0^{\pi/6}$

  $= \frac{1}{2}\sin^2(\pi/6) = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{8}$

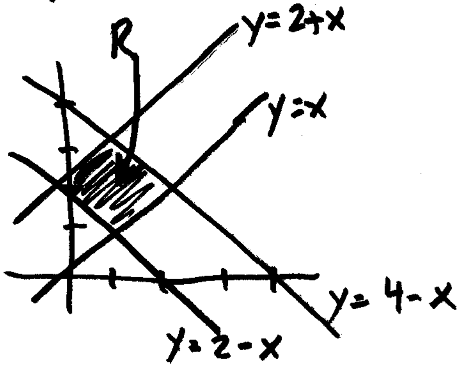
- $\displaystyle\int_0^{2\pi} 1\, d\theta = \theta\Big|_0^{2\pi} = 2\pi$

FINAL: $\frac{1}{5}\cdot 2\pi \cdot \frac{1}{8} = \frac{\pi}{20}$

7. Set up the integral by making an appropriate change of variables for the integral $\int\int_R \frac{x-y}{x+y}\, dA$ where R is the square enclosed by the lines $y = x$, $y = 2+x$, $y = 4-x$, and $y = 2 - x$. Use the transformation $u = x - y$ and $v = x + y$.

(1) $\frac{x-y}{x+y} = \frac{u}{v}$

(2)



(3) $\begin{cases} x-y=u \\ x+y=v \end{cases}$

$2x = u+v$

$\boxed{x = \frac{1}{2}(u+v)}$

$\begin{cases} x-y=u \\ -x-y=-v \end{cases}$

$-2y = u-v$

$\boxed{y = \frac{1}{2}(v-u)}$

(4) $|J| = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \left|\frac{+1}{2}\right| = \frac{1}{2}$

(5) TRANSFORMATION

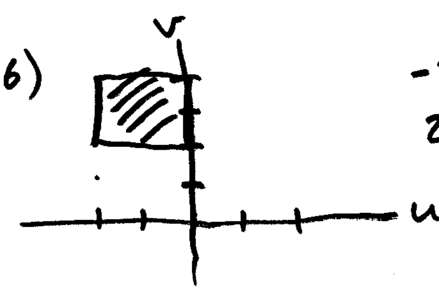
$y = x \rightarrow x-y=0$ $\boxed{u=0}$

$y = 2+x \rightarrow x-y=-2$ $\boxed{u=-2}$

$y = 4-x \rightarrow x+y=4$ $\boxed{v=4}$

$y = 2-x \rightarrow x+y=2$ $\boxed{v=2}$

(6)



$-2 \le u \le 0$

$2 \le v \le 4$

(7) ANSWER: $\int_{-2}^{0}\int_{2}^{4} \frac{u}{v} \cdot \frac{1}{2}\, dv\,du$