EXAM II MATH 232, CALCULUS III, FALL 2016 NAME: SOLUTION

Show all work to receive full credit.

1. (16 points) The acceleration of a football after being thrown due to air resistance and gravity is given by $a(t) = 1i - 2j - e^t k = \langle 1, -2, -e^t \rangle$. Suppose the initial velocity is $\vec{v}(0) = 0i + 50j + 25k$. Find the velocity function $v(\vec{t})$.

- $\vec{v}(t) = \int 1i - 2j - e^t k\, dt = ti - 2tj - e^t k + C$

- $\vec{v}(0) = 0i + 50j + 25k = 0i - 2(0)j - e^0 k + C$

$$\Rightarrow \quad 50j + 25k = -k + C$$

$$\Rightarrow \quad 50j + 26k = C$$

- $\vec{v}(t) = ti - 2tj - e^t k + 50j + 26k$

$$\vec{v}(t) = ti + (50-2t)j + (-e^t + 26)k$$

2. (8 points) Consider the function defined by $\vec{r}(t) = \langle 2\sin(t), -3t, -2\cos(t) \rangle$. Find the arc length of the curve over the interval $0 \le t \le 2$.

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_0^2 \sqrt{(2\cos t)^2 + (-3)^2 + (2\sin t)^2}\, dt$$

$$= \int_0^2 \sqrt{4\cos^2 t + 9 + 4\sin^2 t}\, dt$$

$$= \int_0^2 \sqrt{4(\cos^2 t + \sin^2 t) + 9}\, dt$$

$$= \int_0^2 \sqrt{13}\, dt = \sqrt{13}\, t \Big|_0^2 = 2\sqrt{13}$$

3. (8 points) Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{3x^2+3y^2} = \lim_{r\to 0} \frac{5r^2\cos^2\theta \cdot r\sin\theta}{3r^2\cos^2\theta + 3r^2\sin^2\theta}$$

by first converting to polar coordinates.

$x = r\cos\theta$
$y = r\sin\theta$

$$= \lim_{r\to 0} \frac{5r^3\cos^2\theta\sin\theta}{3r^2}$$

$$= \lim_{r\to 0} \frac{5r}{3} \cdot \cos^2\theta\sin^2\theta = 0 \cdot (\text{BOUNDED})$$

$$= 0$$

4. (20 points) Consider the function $f(x,y) = e^{2xy} + y\cos(x)$.

(a) Find the linearization of $f(x,y)$ at the point $(0,1,2)$.

- $f_x = e^{2xy} \cdot 2y + y(-\sin x)$; $f_x(0,1) = e^{2(0)(1)} \cdot 2(1) + 1(-\sin 0)$
$= 2$

- $f_y = e^{2xy} \cdot 2x + \cos(x)$; $f_y = e^0 \cdot 2(0) + \cos(0) = 0 + 1 = 1$

- FORMULA: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z - 2 = 2(x-0) + 1(y-1)$

$z = 2x + y - 1 + 2$

$z = 2x + y + 1$

(b) Use (a) to approximate $f(0.1, 0.9)$.

$f(0.1, .9) \approx L(.1, .9) = 2(.1) + .9 + 1$
$= 2.1$

5. (16 points) If $z = e^y \sin(x)$, where $x = t \ln s$ and $y = s^2 e^{t-1}$. Using the chain rule, find $\dfrac{\partial z}{\partial s}$ when $s = 1$ and $t = 1$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$x = 1\cdot\ln 1 = 0$

$y = 1^2 e^0 = 1$

$$\frac{\partial z}{\partial x} = e^y\cos(x) \qquad \frac{\partial x}{\partial s} = \frac{t}{s}$$

$$\frac{\partial z}{\partial y} = e^y\sin(x) \qquad \frac{\partial y}{\partial s} = 2se^{t-1}$$

$$\frac{\partial z}{\partial s} = e^y\cos(x)\cdot\frac{t}{s} + e^y\sin(x)\cdot 2se^{t-1}$$

$$\left.\frac{\partial z}{\partial s}\right|_{t=1,\,s=1} = e^1\cos(0)\cdot\frac{1}{1} + e^1\sin(0)\cdot 2(1)e^0 = e + 0 = \boxed{e}$$

6. (12 points) Let $f(x, y) = x^2y^2 + xy - 1$.

(a) Find the gradient of f.

$$\nabla f = \langle f_x, f_y\rangle = \langle 2xy^2 + y,\ 2x^2y + x\rangle$$

(b) Find the rate of change of f in the direction of $\vec{v} = \langle 3, 4\rangle$ at the point $P(1, 0, -1)$.

$$D_{\vec{v}}f(1,0) = \nabla f(1,0)\cdot\frac{\vec{v}}{|\vec{v}|} \quad ; \quad \nabla f(1,0) = \langle 0, 1\rangle$$

$$= \frac{\langle 0,1\rangle\cdot\langle 3,4\rangle}{5}$$

$$= \frac{0(3) + 1(4)}{5}$$

$$= \frac{4}{5}$$

7. (20 points) Find the local maximum, local minimum, and saddle points of the function $f(x,y) = x^3 - y^2 - xy$.

i) $f_x = 3x^2 - y$ $\qquad$ $3x^2 - y = 0 \rightarrow y = 3x^2$

ii) $f_y = -2y - x$ $\qquad$ $-2y - x = 0$

PLUG $y = 3x^2$ INTO $-2y - x = 0$

$$-6x^2 - x = 0$$

$$-x(6x+1) = 0$$

$$x = 0, \quad x = -1/6$$

IF $x = 0$, $y = 3(0)^2 = 0$

IF $x = -1/6$, $y = 3(-1/6)^2 = \frac{1}{12}$

CRITICAL POINTS

$(0,0)$, $(-1/6, 1/12)$

iii) $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -1 \\ -1 & -2 \end{vmatrix} = 6x(-2) - (-1)^2$
$= -12x - 1$

$(0,0)$: $D(0,0) = -12(0) - 1 = -1 < 0$ SO $(0,0)$ IS A SADDLE POINT

$(-1/6, 1/12)$: $D(-1/6, 1/12) = -12(-1/6) - 1 = 2 - 1 = 1 > 0$

AND $f_{xx}(-1/6, 1/12) = 6(-1/6) = -1 < 0$ SO $(-1/6, 1/12)$ IS A LOCAL MAXIMUM