EXAM 3 MATH 230, CALCULUS II NAME: Solution

Show all work to receive full credit.

1. Suppose that $0 \le a_n \le b_n$ for all n. Mark each of the following statements clearly as TRUE or FALSE. If the answer is FALSE, given an example showing why.

   (a) If $\sum a_n$ diverges, then $\sum b_n$ must diverge.

   True (Direct Comparison Test)

   (b) If $\sum a_n$ diverges, then $\sum a_n^2$ must diverge.

   False. Consider $a_n = \frac{1}{n}$. $\sum \frac{1}{n}$ diverges but $\sum \frac{1}{n^2}$ converges

   (c) If $\sum b_n$ converges, then $\sum (a_n + b_n)$ must converge.

   True. Uses the Direct Comparison Test again.

   (d) If $\sum a_n$ diverges, then $\sum (-1)^n a_n$ must diverge.

   False. Consider $\sum \frac{1}{n}$. $\sum \frac{1}{n}$ diverges but $\sum (-1)^n \cdot \frac{1}{n}$ converges by AST.

   (e) If $\sum a_n$ converges, then $\sum (3 + \cos n) a_n$ must converge.

   True. Note: $\sum 2a_n < \sum (3 + \cos n) a_n < \sum 4 a_n$

   (f) If $a_n \to 0$ then $\sum a_n$ must converge.

   False: $a_n = \frac{1}{n} \to 0$ but $\sum \frac{1}{n}$ diverges

2. Consider the series

$$\sum_{n=1} a_n = 5 - \frac{5}{4} + \frac{5}{9} - \frac{5}{16} + \ldots$$

(a) Find a formula for a_n.

$$a_n = \frac{5 \cdot (-1)^{n-1}}{n^2} \text{ or } (-1)^{n-1} \cdot \frac{5}{n^2}$$

$$\Rightarrow \sum_{n=1} (-1)^{n-1} \cdot \frac{5}{n^2}$$

(b) Show that the series converges. Let $b_n = \frac{5}{n^2}$.

(1) b_n is positive ✓

(2) b_n is decreasing ✓ Hint: Show $(b_n)' < 0$ for $n > 1$

(3) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{5}{n^2} = 0$ ✓

By the Alternating Series Test, $\sum_{n=1} (-1)^{n-1} \cdot \frac{5}{n^2}$ converges.

(c) Find a value of n such that the nth partial sum S_n approximates the sum of the series to within 0.001. For an Alternating Series, we know

$$|R_n| \le b_{n+1}$$

- Find n so that $b_{n+1} < .001$
- $b_{n+1} = \frac{5}{(n+1)^2} < .001 \rightarrow \frac{5}{.001} < (n+1)^2$

$$\sqrt{\frac{5}{.001}} < n+1 \longrightarrow \sqrt{\frac{5}{.001}} - 1 < n$$

$$n > 69.71$$

So we need to find S_{70}

3. Evaluate $\int_2^\infty \frac{x}{(1+x^2)^2}\, dx$

- Let $u = 1+x^2$, $du = 2x\,dx \longrightarrow \frac{1}{2}du = x\,dx$

- If $x \to \infty$, $u \to \infty$

  If $x = 2$, $u = 5$

- $\int_5^\infty \frac{1}{2} \cdot \frac{1}{u^2}\, du = \int_5^\infty \frac{1}{2} u^{-2} du = \lim_{t\to\infty} \int_5^t \frac{1}{2} u^{-2} du$

$= \lim_{t\to\infty} \left[\frac{-1}{2u} \Big|_5^t \right] = \lim_{t\to\infty} \left[\frac{-1}{2t} + \frac{1}{10} \right] = 0 + \frac{1}{10} = \underline{\underline{\frac{1}{10}}}$

4. Does $\int_1^\infty \frac{\ln x}{x^2}\, dx$ converge or diverge? Justify your answers.

- Use By Parts. Ignore bounds for now

- $u = \ln x$ $\qquad dv = \frac{1}{x^2} dx$

  $du = \frac{1}{x} dx$ $\qquad v = \frac{-1}{x}$

- $uv - \int v\,du \rightarrow \frac{-1}{x}\ln x - \int \frac{1}{x} \cdot \frac{-1}{x}\, dx$

$= \frac{-1}{x}\ln x + \int \frac{1}{x^2} dx$

$= \frac{-1}{x}\ln x - \frac{1}{x}$

$\Rightarrow \frac{-1}{x}\ln x - \frac{1}{x} \Big|_1^\infty = \lim_{t\to\infty} \left[\frac{-1}{x}\ln x - \frac{1}{x} \Big|_1^t \right] = \lim_{t\to\infty} \left[\frac{-\ln x - 1}{x} \Big|_1^t \right]$

$= \lim_{t\to\infty} \left[\frac{-\ln t - 1}{t} - \frac{\ln 1 - 1}{1} \right] = \lim_{t\to\infty} \frac{-\ln t - 1}{t} - \lim_{t\to\infty} \frac{\ln 1 - 1}{1}$

$= 0 + 1 = 1$ Conv. by Integral Test

5. Determine whether each of the following series diverges, converges conditionally, or converges absolutely. Justify your answers with tests.

(a) $\sum \frac{\sqrt{n^4+1}}{5n^2-4}$ Acts like $\frac{\sqrt{n^4}}{5n^2} = \frac{n^2}{5n^2} = \frac{1}{5}$ when n is large.

Check for divergence:

$$\lim_{n\to\infty} \frac{\sqrt{n^4+1}}{5n^2-4} = \lim_{n\to\infty} \frac{\sqrt{n^4(1+1/n^4)}}{5n^2-4} = \lim_{n\to\infty} \frac{n^2\sqrt{1+1/n^4}}{5n^2-4} \leftarrow \text{Mult. by } \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+1/n^4}}{5-4/n^2} = \frac{\sqrt{1}}{5} \neq 0.$$ So $\sum \frac{\sqrt{n^4+1}}{5n^2-4}$ diverges by the Divergence Test

(b) $\sum_{1}^{\infty} \frac{(-1)^n \ln n}{n^2}$ Let $b_n = \frac{\ln n}{n^2}$.

(1) b_n is positive ✓

(2) b_n is decreasing ✓ Show $(b_n)' < 0$ for large n

(3) $\lim_{n\to\infty} \frac{\ln n}{n^2} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = 0$ ✓ So $\sum \frac{(-1)^n \ln n}{n^2}$ converges

Now check for Absolute: $\sum \left| \frac{(-1)^n \ln n}{n^2} \right| = \sum \frac{\ln n}{n^2}$. You already showed this converges in #4. Therefore $\sum \frac{\ln n}{n^2}$ converges absolutely.

(c) $\sum \frac{n^5 4^{n+2}}{(-3)^n}$

Try Ratio Test: $a_n = \frac{n^5(4^{n+2})}{(-3)^n}$, $a_{n+1} = \frac{(n+1)^5 \cdot 4^{n+3}}{(-3)^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5 \cdot 4^{n+3}}{(-3)^{n+1}} \cdot \frac{(-3)^n}{n^5 \cdot 4^{n+2}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^5}{n^5} \cdot \frac{4^n \cdot 4^3}{4^n \cdot 4^2} \cdot \frac{(-3)^n}{(-3)^n \cdot (-3)} \right|$$

$$= \lim \left| \frac{(n+1)^5}{n^5} \cdot \frac{4^3}{4^2} \cdot \frac{1}{(-3)} \right| = \frac{4}{3} > 1.$$

By the Ratio test, $\sum \frac{n^5 4^{n+2}}{(-3)^n}$ diverges