Name: Solution

Math 230
Section 1
Fall 2013

Exam 2

Show all your work to receive full credit.

1. Differentiate. [6 pts each]

a) $f(x) = x^{\sin x} = e^{\sin x \cdot \ln x}$

$$f'(x) = e^{\sin x \cdot \ln x} \cdot (\sin x \cdot \ln x)'$$

$$= e^{(\sin x \cdot \ln x)} \cdot \left(\sin x \cdot \frac{1}{x} + \cos x \cdot \ln x\right)$$

$$= e^{\sin x \ln x}\left(\frac{\sin x}{x} + \cos x \cdot \ln x\right)$$

b) $g(t) = 10^t - e^t + t^e - e^\pi$

$$g'(t) = 10^t \ln 10 - e^t + e t^{e-1} - 0$$

c) $y = \log_7(3x)\arcsin(2x)$ Product Rule

$$y' = \log_7(3x) \cdot \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 + \frac{1}{3x \ln 7} \cdot 3 \cdot \sin^{-1}(2x)$$

$$= \frac{2\log_7(3x)}{\sqrt{1-(2x)^2}} + \frac{\sin^{-1}(2x)}{x \ln 7}$$

d) $k(r) = 3^{\arctan(5r)}$

$$k'(r) = 3^{\tan^{-1}(5r)} \cdot \ln(3) \cdot \frac{1}{1+(5r)^2} \cdot 5$$

$$= \frac{5\ln(3) \cdot 3^{\tan^{-1}(5r)}}{1+25r^2}$$

2. Perform a trig substitution to rewrite the integral in terms of trigonometric functions. Do not evaluate the trig integrals. [7 pts each]

a) $\int x^3\sqrt{1-x^2}\,dx$

Let $x = \sin(\theta)$, $dx = \cos(\theta)d\theta$

$$\int \sin^3\theta \sqrt{1-\sin^2\theta} \cdot \cos(\theta)d\theta$$

$$= \int \sin^3\theta \sqrt{\cos^2\theta}\, \cos\theta\, d\theta = \int \sin^3\theta \cos^2\theta d\theta$$

b) $\int \frac{\sqrt{9x^2-25}}{x}\,dx = \int \frac{\sqrt{(3x)^2-25}}{x}dx$. Let $x = \frac{5}{3}\sec\theta$, $dx = \frac{5}{3}\sec\theta\tan\theta d\theta$

$$\int \frac{\sqrt{(3\cdot 5/3\sec\theta)^2-25}}{5/3\sec\theta} \cdot \frac{5}{3}\sec\theta\tan\theta d\theta = \int \frac{\sqrt{25\sec^2\theta-25}}{1} \cdot \tan\theta\, d\theta$$

$$= \int \frac{\sqrt{25(\sec^2\theta-1)}}{1} \cdot \tan\theta d\theta = \int \sqrt{25\tan^2\theta}\; \tan\theta d\theta = \int 5\tan^2\theta\, d\theta$$

3. Evaluate. [7 pts each]

1. $\int x3^x\,dx$. By Parts. let $u = x$, $dv = 3^x dx$
$du = dx$, $v = \frac{3^x}{\ln 3}$

$$= \frac{x3^x}{\ln 3} - \int \frac{3^x}{\ln 3}dx = \frac{x3^x}{\ln 3} - \frac{3^x}{(\ln 3)^2} + C$$

2. $\int_1^e x^4 \ln x\,dx$. By Parts Let $u = \ln x$, $dv = x^4 dx$
$du = \frac{1}{x}dx$, $v = \frac{1}{5}x^5$

$$\int x^4 \ln x\, dx = \frac{1}{5}x^5\ln x - \int \frac{1}{5}x^5 \cdot \frac{1}{x}dx = \frac{1}{5}x^5\ln x - \int \frac{1}{5}x^4 dx$$

$$= \frac{1}{5}x^5\ln x - \frac{1}{25}x^5 + C$$

4. Find the limit, if it exists. [6 pts each]

1. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty$. Rewrite as $\lim\limits_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$. Focus on $x\ln(1+\frac{1}{x})$

- $\lim\limits_{x\to\infty} x\ln(1+\frac{1}{x}) = \infty\cdot\ln 1 = \infty\cdot 0$. Rewrite as $\lim\limits_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{\frac{1}{x}}$

$\stackrel{LH}{=} \lim\limits_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = 1$ Final answer: $e^1 = e$

2. $\lim\limits_{x\to 2}\frac{e^x-e^2}{xe^x-e^x-e^2} = \frac{e^2-e^2}{2e^2-e^2-e^2} = \frac{0}{0}$. Use LH

$\stackrel{LH}{=} \lim\limits_{x\to 2}\frac{e^x}{xe^x+1\cdot e^x-e^x} = \lim\limits_{x\to 2}\frac{e^x}{xe^x} = \lim\limits_{x\to 2}\frac{1}{x} = \frac{1}{2}$

5. Consider the sequence $\{-9, 6, -4, \frac{8}{3}, \ldots\}$. Find a formula for the general term a_n, assuming the pattern continues. [6 pts] . Assume the sequence is geometric: $\{a, ar, ar^2, ar^3, \ldots\}$

$a_n = ar^n$

Use ↓ to find a & r

- $a = -9$
- $ar = +6$
- $ar^2 = -4$

$ar = +6$

$-9r = +6$

$r = -2/3$

So $a_n = -9(-2/3)^n$

Check a_0, a_1, a_2, a_3 to verify

6. Determine if the sequence converges. If so, find the limit. [6 pts each]

1. $\{\ln(n+1)-\ln n\}_{n=1}^{\infty} = \left\{\ln\left(\frac{n+1}{n}\right)\right\}_{n\geq 1}^{\infty}$

$\lim\limits_{n\to\infty}\ln\left(\frac{n+1}{n}\right) = \ln(1) = 0$

2. $\left\{\frac{\sqrt{4n^6-17}}{n^3-5n}\right\}_{n=1}^{\infty}$ $\frac{\sqrt{n^6(4-\frac{17}{n^6})}}{n^3-5n} = \frac{n^3\sqrt{4-\frac{17}{n^6}}}{n^3-5n} = \frac{\sqrt{4-\frac{17}{n^6}}}{1-\frac{5}{n^2}}$

$\lim\limits_{n\to\infty}\frac{\sqrt{4-17/n^6}}{1-5/n^2} = \frac{\sqrt{4-0}}{1-0} = \frac{\sqrt{4}}{1} = 2$

7. Evaluate. [7 pts each]

a) $\int \frac{\sin^2\theta}{\cos\theta}\,d\theta = \int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta = \int \frac{1}{\cos\theta} - \cos\theta \, d\theta = \int \sec\theta - \cos\theta\, d\theta$

$= \ln\left|\sec\theta + \tan\theta\right| - \sin\theta + C$

b) $\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cdot \cos x\,dx = \int \sin^3 x(1-\sin^2 x)\cos x\,dx$

Let $u = \sin x$, $du = \cos x\,dx \longrightarrow \int u^3(1-u^2)\,du$

$= \int u^3 - u^5\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$

$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C$

c) $\int \cos^4 t\,dt = \int (\cos^2 t)(\cos^2 t)\,dt = \int \frac{1}{2}(1+\cos(2t)) \cdot \frac{1}{2}(1+\cos(2t))\,dt$

$= \frac{1}{4}\int 1 + 2\cos(2t) + \cos^2(2t)\,dt = \frac{1}{4}\int 1 + 2\cos(2t) + \frac{1}{2}(1+\cos(4t))\,dt$

$= \frac{1}{4}\int 1 + 2\cos(2t) + \frac{1}{2} + \frac{1}{2}\cos(4t)\,dt = \frac{1}{4}\int \frac{3}{2} + 2\cos(2t) + \frac{1}{2}\cos(4t)\,dt$

$= \frac{1}{4}\left(\frac{3}{2}t + \sin(2t) + \frac{1}{8}\sin(4t)\right) + C$ Also try reduction formula

d) $\int \frac{dy}{(\ln 3)(y)(1+(\log_3 y)^2)}$

Let $u = \log_3 y$, $du = \frac{1}{\ln 3 \cdot y}\,dy \longrightarrow \int \frac{1}{1+u^2}\,du$

$= \tan^{-1}(u) + C = \tan^{-1}(\log_3 y) + C$