Name: Solution

Math 230
Section 1
Fall 2013

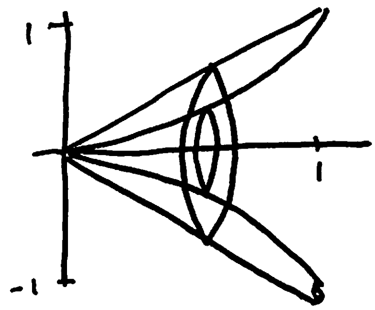
Exam 1

Show all your work to receive full credit.

1. Sketch the region enclosed by the given curves, and set up an integral for the volume of the solid obtained by rotating the region about the specified line. [5 pts each]

a) $y = x$, $y = x^4$; about the x-axis

Washer

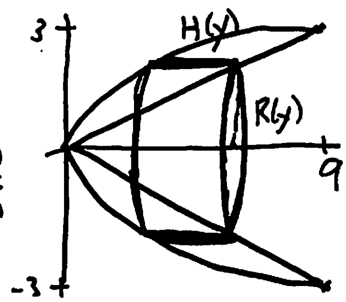


$R(x) = x$
$r(x) = x^4$
Bounds:
$a = 0$
$b = 1$

$$V = \int_0^1 \pi R(x)^2 - \pi r(x)^2 \, dx$$
$$= \int_0^1 \pi x^2 - \pi (x^4)^2 \, dx$$
$$= \pi \int_0^1 x^2 - x^8 \, dx$$

b) $x = 3y$, $x = y^2$; about the x-axis

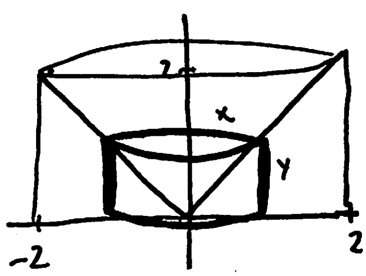
Shell



$$\int_0^3 2\pi y(3y - y^2)\, dy$$
$$\int_c^d 2\pi R(y) H(y)\, dy$$

c) $y = x$, $x = 2$; about the y-axis

Shell



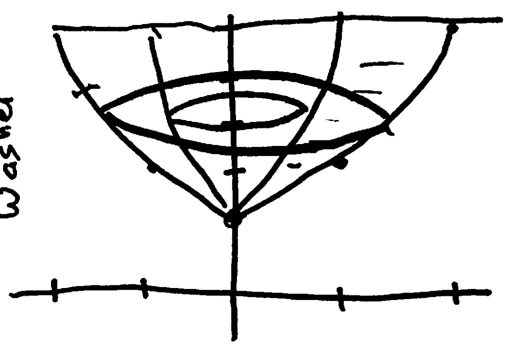
Shell: $\int_0^2 2\pi(x)(x)\, dx = \int_0^2 2\pi x^2 \, dx$

washer: $\int_0^2 \pi(2)^2 - \pi(y)^2 \, dy$

d) (in the first quadrant) $y = x^2 + 1$, $y = 4x^2 + 1$, $y = 5$; about the y-axis

$y = x^2 + 1 \rightarrow x = \sqrt{y-1}$
$y = 4x^2 + 1 \rightarrow x = \sqrt{\frac{y-1}{4}}$

Washer



$$\int_1^5 \pi\left(\sqrt{y-1}\right)^2 - \pi\left(\sqrt{\frac{y-1}{4}}\right)^2 dy$$

2. Let f be a differentiable, invertible function. Given the following, find $(f^{-1})'(3)$. [6 pts]

$f(3) = 5 \quad f(2) = 3$
$f'(3) = 7 \quad f'(2) = 4$

$$(f^{-1})'(3) = \frac{1}{f'(\underbrace{f^{-1}(3)}_{*})} = \frac{1}{f'(2)} = \frac{1}{4}$$

$*$ If $f(2) = 3$, then $f^{-1}(3) = 2$

3. Write the midpoint rule approximation for the given integral with $n = 4$. [6 pts]

$\int_1^3 e^{x^2}\,dx$, • $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$, • $x_i = a + i\Delta x = 1 + i \cdot \frac{1}{2}$

• $x_0 = 1 + 0 \cdot \frac{1}{2} = 1$, $x_1 = 1 + 1 \cdot \frac{1}{2} = 1.5$, $x_2 = 1 + 2 \cdot \frac{1}{2} = 2$, $x_3 = 2.5$, $x_4 = 3$

Midpoint x-values → • $x_1^* = 1.25$, $x_2^* = 1.75$, $x_3^* = 2.25$, $x_4^* = 2.75$

Formula: $M_4 = \Delta x\left[f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)\right]$

$$= \frac{1}{2}\left[e^{1.25^2} + e^{1.75^2} + e^{2.25^2} + e^{2.75^2}\right] = 1054.3939$$

4. Differentiate [6 pts each]

a) $f(x) = (\ln(2x+1))^3$

$$f'(x) = 3\left(\ln(2x+1)\right)^2 \cdot \frac{1}{2x+1} \cdot 2 = \frac{6\left(\ln(2x+1)\right)^2}{2x+1}$$

b) $g(t) = t^e - \ln(\sin t)$

$$g'(t) = et^{e-1} - \frac{1}{\sin(t)} \cdot \cos(t) = et^{e-1} - \frac{\cos(t)}{\sin(t)}$$

c) $h(s) = \underbrace{(3s+1)e^{s^2}}_{\text{product rule}} + e^{\ln s}$ → Note: $e^{\ln s} = s$

$$h'(s) = (3s+1)e^{s^2} \cdot 2s + 3e^{s^2} + 1$$

$$= 2s(3s+1)e^{s^2} + 3e^{s^2} + 1$$

5. Set up an integral that represents the length of the curve $x = \frac{y^3}{3} + \frac{1}{4y}$, $1 \le y \le 2$. [5 pts]

(1) Formula: $L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_1^2 \sqrt{\left(y^2+\frac{1}{4}y^{-2}\right)^2}\,dy = \int_1^2 y^2+\frac{1}{4}y^{-2}\,dy$

(2) $x = \frac{y^3}{3}+\frac{1}{4}y^{-1}$

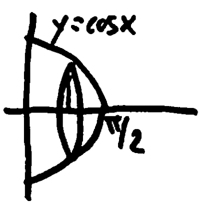
(3) $\frac{dx}{dy} = y^2 - \frac{1}{4}y^{-2}$

(4) $\left(y^2-\frac{1}{4}y^{-2}\right)^2 = y^4 - \frac{1}{2} + \frac{1}{16}y^{-4}$

(5) $1+\left(\frac{dx}{dy}\right)^2 = 1+\left(y^4-\frac{1}{2}+\frac{1}{16}y^{-4}\right) = y^4+\frac{1}{2}+\frac{1}{16}y^{-4} = \left(y^2+\frac{1}{4}y^{-2}\right)^2$

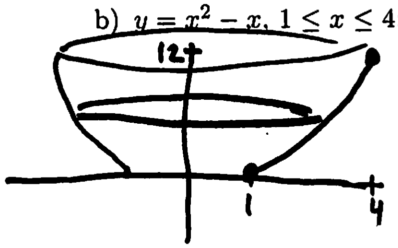
6. Set up an integral that represents the area of the surface obtained by rotating the given curve about the specified axis. [5 pts each]

a) $y = \cos x$, $0 \le x \le \frac{\pi}{2}$; about the x-axis



$$\int_0^{\pi/2} 2\pi\cos(x)\sqrt{1+(-\sin x)^2}\,dx$$

b) $y = x^2 - x$, $1 \le x \le 4$; about the y-axis



$$\int_1^4 2\pi x\sqrt{1+(2x-1)^2}\,dx$$

c) $x = 2\sqrt{y^3}$, $2 \le y \le 3$; about the x-axis

$\frac{dx}{dy} = 3y^{1/2}$

$x = 2y^{3/2}$

$$\int_2^3 2\pi y\sqrt{1+\left(3y^{1/2}\right)^2}\,dy$$

7. Let $y = \frac{(x-3)^2(2x-1)^4}{(x+1)^3}$. Find $\frac{dy}{dx}$. [6 pts]

$\ln y = \ln\left(\frac{(x-3)^2(2x-1)^4}{(x+1)^3}\right)$

$\ln y = 2\ln(x-3)+4\ln(2x-1)-3\ln(x+1)$

Differentiate:

$\frac{1}{y}\cdot y' = \frac{2}{x-3}+\frac{4}{2x-1}\cdot 2 - \frac{3}{x+1} \longrightarrow y' = y\left(\frac{2}{x-3}+\frac{8}{2x-1}-\frac{3}{x+1}\right)$

$y' = \frac{(x-3)^2(2x-1)^4}{(x+1)^3}\left(\frac{2}{x-3}+\frac{8}{2x-1}-\frac{3}{x+1}\right)$

8. Evaluate [6 pts each]

a) $\int \frac{5-x}{x}\,dx = \int \frac{5}{x} - \frac{x}{x}\,dx = \int \frac{5}{x} - 1\,dx$

Integrate: $\int \frac{5}{x} - 1\,dx = 5\ln(x) - x + C$

b) $\int e^t\sqrt{1+e^t}\,dt$

(1) Let $u = e^t$

(2) $du = e^t\,dt$

(3) Substitute

$$\int e^t\sqrt{1+e^t}\,dt = \int \sqrt{u}\,du$$
$$= \int u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}(1+e^t)^{3/2} + C$$

c) $\int \frac{1}{x\ln x}\,dx$

(1) let $u = \ln x$

(2) $du = \frac{1}{x}dx$

(3) Substitute

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + C$$
$$= \ln\left|\ln x\right| + C$$

d) $\int_0^1 y^2 e^{3y^3}\,dy$

(1) let $u = 3y^3$

(2) $du = 9y^2dy \rightarrow \frac{1}{9}du = y^2dy$

(3) Bounds:

If $y=0$, $u = 3(0)^3 = 0$

If $y=1$, $u = 3(1)^3 = 3$

(4) Substitute

$$\int_0^3 \frac{1}{9}e^u\,du$$
$$= \frac{1}{9}e^u\Big|_0^3$$
$$= \frac{1}{9}e^3 - \frac{1}{9}e^0$$
$$= \frac{1}{9}(e^3 - 1)$$