Math 230, RED, Spring 2009, Exam 2 NAME Solution

1. (5 points each) Compute the following integrals.

(a) $\int_1^{e^3} x(\ln x)^2\,dx$ $\quad u = (\ln x)^2 \quad dv = x\,dx$

$du = 2(\ln x)\cdot\frac{1}{x} \quad v = \frac{x^2}{2}$

$\frac{x^2}{2}(\ln x)^2 - \int x \ln x\,dx$

$u = \ln x,\ dv = x\,dx$

$du = \frac{1}{x}dx,\ v = \frac{x^2}{2}$

$\frac{x^2}{2}(\ln x)^2 - \left(\frac{x^2}{2}\ln x - \int \frac{x}{2}\,dx\right)$

$\frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4}\Big|_1^{e^3}$

$\left(\frac{e^6}{2}\cdot(\ln e^3)^2 - \frac{e^6}{2}\ln(e^3) + \frac{e^6}{4}\right) - \left(\frac{1}{4}\right)$

$\frac{13}{4}e^6 - \frac{1}{4}$

(b) $\int \tan^3(4x)\,dx$. let $u = 4x \quad du = 4dx \rightarrow \frac{1}{4}du = dx$

$\frac{1}{4}\int \tan^3 u\,du = \frac{1}{4}\int(\sec^2 u - 1)\tan u\,du = \frac{1}{4}\int \sec^2 u \tan u\,du - \frac{1}{4}\int \tan u\,du$

$= \frac{1}{4}\tan u - \frac{1}{4}\ln|\sec u| + C$

$= \frac{1}{4}\tan(4x) - \frac{1}{4}\ln|\sec(4x)| + C$

(c) $\int \cos^5(3x)\,dx = \int \cos^4(3x)\cos(3x)\,dx = \int(\cos^2(3x))^2\cos(3x)\,dx$

$= \int(1-\sin^2(3x))^2\cos(3x)\,dx$. let $u = \sin(3x)$, $du = 3\cos(3x)dx$ or $\frac{1}{3}du = \cos(3x)dx$

$= \frac{1}{3}\int(1-u^2)^2\cdot du = \frac{1}{3}\int 1 - 2u^2 + u^4\,du$

$= \frac{1}{3}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) = \frac{1}{3}\left(\sin(3x) - \frac{2}{3}\sin^3(3x) + \frac{1}{5}\sin^5(3x)\right) + C$

(d) $\int_0^{\pi/3} \sin^2(3x)\,dx$

$= \int_0^{\pi/3} \frac{1}{2} - \frac{1}{2}\cos(6x)\,dx = \frac{1}{2}x - \frac{1}{12}\sin(6x)\Big|_0^{\pi/3}$

$= \left(\frac{1}{2}\cdot\frac{\pi}{3} - \frac{1}{12}\sin(2\pi)\right) - \left(\frac{1}{2}\cdot 0 - \frac{1}{12}\sin(6\cdot 0)\right)$

$= \frac{\pi}{6}$

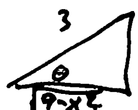
(e) $\int \frac{\sec^{-1} x}{x\sqrt{x^2-1}}\, dx$ Let $u = \sec^{-1} x$, $du = \frac{1}{x\sqrt{x^2-1}}\, dx$

$$\Rightarrow \int u\, du = \frac{u^2}{2} + C = \frac{[\sec^{-1}(u)]^2}{2} + C$$

(f) $\int \frac{4x\, dx}{\sqrt{9-x^2}}$ * Best way is to do a u-sub: $u = 9-x^2$, $du = -2x\,dx$. But I'm going to do a trig sub, just because...

$x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$ $\rightarrow \int \frac{4\cdot 3\sin\theta \cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{36\sin\theta\cos\theta\, d\theta}{\sqrt{9\cos^2\theta}}$

$$= \int \frac{36\sin\theta\,\cancel{\cos\theta}\,d\theta}{3\,\cancel{\cos\theta}} = \int 12\sin\theta\, d\theta = -12\cos\theta + C$$

Use 

$\rightarrow -12\cos\theta = -12\left(\frac{\sqrt{9-x^2}}{3}\right) = -4\sqrt{9-x^2} + C$

Check answer with u-sub

(g) $\int \frac{dx}{x + x(\ln x)^2}$

$= \int \frac{dx}{x(1+(\ln x)^2)}$. Let $u = \ln x$, $du = \frac{1}{x}\, dx$

$$= \int \frac{1}{1+u^2}\, du = \tan^{-1}(u) + C = \tan^{-1}(\ln x) + C$$

(h) $\int_1^{\sqrt{3}} \frac{2^{\tan^{-1} x}}{1+x^2}$ let $u = \tan^{-1} x$, $du = \frac{1}{1+x^2}\,dx$. If $x = 1$, $u = \tan^{-1}(1) = \pi/4$

If $x = \sqrt{3}$, $u = \tan^{-1}(\sqrt{3}) = \pi/3$

$$\Rightarrow \int_{\pi/4}^{\pi/3} 2^u\, du = \frac{2^u}{\ln 2}\Big|_{\pi/4}^{\pi/3}$$

$$= \frac{2^{\pi/3}}{\ln 2} - \frac{2^{\pi/4}}{\ln 2}$$

2. (5 points each) Compute the following limits. No work need be shown, but you will lose points for writing false arguments.

(a) $\lim_{x\to\infty}\left(e^{-3x+2}+\tan^{-1}(x)\right)$ $= e^{-\infty} + \tan^{-1}(-\infty) = 0 - \pi/2 = -\pi/2$

(b) $\lim_{n\to\infty}\left(1-\frac{5}{3n}\right)^{n/5}$ Rewrite: $\lim_{n\to\infty} e^{n/5\ln\left(1-\frac{5}{3n}\right)}$. Do $\lim_{n\to\infty} \frac{n}{5}\ln\left(1-\frac{5}{3n}\right)$

$= \lim_{n\to\infty} \frac{\ln\left(1-\frac{5}{3n}\right)}{\frac{5}{n}} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1-\frac{5}{3n}}\cdot\frac{5}{3n^2}}{\frac{-5}{n^2}} = \lim_{n\to\infty}\frac{1}{1-\frac{5}{3n}}\cdot\frac{-1}{3} = -\frac{1}{3}$

Final Answer: $e^{-1/3}$

(c) $\lim_{n\to\infty}\frac{(\ln n)^3}{n^{1/15}}$ $= 0$

Do LH a few times. Simplify after each LH.

(d) $\lim_{x\to 0}(\ln(\sin x)-\ln(x))$ $= \lim_{x\to 0}\left(\ln\left(\frac{\sin x}{x}\right)\right)$. Note: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

so $\lim_{x\to 0}\ln\left(\frac{\sin x}{x}\right) = \ln(1) = 0$

(e) $\lim_{n\to\infty}\sqrt[n]{5n^2+7}$ $= \lim_{n\to\infty}(5n^2+7)^{1/n} = \lim_{n\to\infty} e^{1/n\ln(5n^2+7)}$. Focus on $\lim_{n\to\infty}\frac{1}{n}\ln(5n^2+7)$

$\lim_{n\to\infty}\frac{\ln(5n^2+7)}{n} \overset{LH}{=} \lim_{n\to\infty}\frac{\frac{1}{5n^2+7}\cdot 10n}{1} = \lim_{n\to\infty}\frac{10n}{5n^2+7} \overset{LH}{=} \lim_{n\to\infty}\frac{10}{10n} = 0$

Final: $e^0 = 1$

(f) $\lim_{x\to 0^+}\left(\frac{1}{\sin(x)-x}\right)$ $= \lim_{x\to 0^+}\frac{1/x}{\frac{\sin x}{x}-1}$ A V.A. occurs at $x=0$.

Note: $\frac{1}{x}>0$ when $x\to 0^+$. $\frac{\sin x}{x}\le 1$ when $x\to 0^+$. So $\frac{\sin x}{x}-1\le 0$ as $x\to 0^+$

Final: $\frac{+}{-}\to -\infty$

(g) $\lim_{x\to\infty}\frac{\sin x}{x}$ (Careful!)

$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$ And $\lim_{x\to\infty}-\frac{1}{x} = \lim_{x\to\infty}\frac{1}{x} = 0$. By Squeeze thm, $\lim_{x\to 0}\frac{\sin x}{x} = 0$

(h) $\lim_{x\to\pi}\frac{\sin^{-1}(x-\pi)}{x-\pi}$ $\overset{LH}{=} \lim_{x\to\pi}\frac{\frac{1}{\sqrt{1-(x-\pi)}}\cdot(1)}{1} = \lim_{x\to\pi}\frac{1}{\sqrt{1-(x-\pi)}} = \frac{1}{\sqrt{1}} = 1$

3. (20 points) Compute the following limits. You will be graded on the correctness and clarity of your arguments.

(a) $\lim_{x\to\infty}\left(e^{3x}+1/x\right)^{1/x}$. $= \lim_{x\to\infty} e^{\frac{1}{x}\ln\left(e^{3x}+\frac{1}{x}\right)}$, Focus on $\lim_{x\to\infty} \frac{1}{x}\ln\left(e^{3x}+\frac{1}{x}\right)$

$$= \lim_{x\to\infty} \frac{\ln\left(e^{3x}+\frac{1}{x}\right)}{x} \overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{e^{3x}+\frac{1}{x}}\cdot\left(3e^{3x}-\frac{1}{x^2}\right)}{1}$$

$$= \lim_{x\to\infty} \frac{3e^{3x}-\frac{1}{x^2}}{e^{3x}+\frac{1}{x}} \cdot \frac{x^2}{x^2} = \lim_{x\to\infty} \frac{3x^2e^{3x}-1}{x^2e^{3x}+x}$$

$$\overset{LH}{=} \lim_{x\to\infty} \frac{3x^2\cdot 3e^{3x}+6xe^{3x}}{x^2\cdot 3e^{3x}+2xe^{3x}} = \lim_{x\to\infty} \frac{3\left(3x^2e^{3x}+2xe^{3x}\right)}{3x^2e^{3x}+2xe^{3x}}$$

$$= \lim_{x\to\infty} 3 = 3 \qquad \text{Final: } e^3$$

(b) $\lim_{x\to 0^-} \frac{\cos x - 1}{e^x - x - 1} \overset{LH}{=} \lim_{x\to 0^-} \frac{-\sin x}{e^x-1}$

$$\overset{LH}{=} \lim_{x\to 0^-} \frac{-\cos x}{e^x}$$

$$= \frac{-\cos(0)}{e^0}$$

$$= \frac{-1}{1}$$

$$= -1$$

4. (20 points) Compute $\int \frac{\sqrt{y^2-25}}{y^3}\,dy$. Be sure to substitute back to the original variable.

Let $y = 5\sec\theta$, $dy = 5\sec\theta\tan\theta\,d\theta$, $y^3 = 5^3\sec^3\theta$

$$\int \frac{\sqrt{25\sec^2\theta - 25}}{5^3\sec^3\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sqrt{25\tan^2\theta}\cdot\tan\theta\,d\theta}{5^2\sec^2\theta}$$

$$= \int \frac{5\tan\theta\cdot\tan\theta}{5^2\sec^2\theta}\,d\theta = \int \frac{1}{5}\tan^2\theta\,\overset{\cos}{\cancel{\sec}}{}^2\theta\,d\theta$$

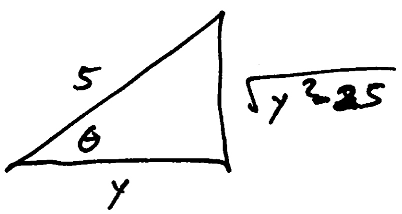
$$= \int \frac{1}{5}\sin^2\theta\,d\theta = \int \frac{1}{5}\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)d\theta = \int \frac{1}{10} - \frac{1}{10}\cos(2\theta)\,d\theta$$

$$= \frac{1}{10}\theta - \frac{1}{20}\sin(2\theta) + C$$

$$= \frac{1}{10}\theta - \frac{1}{20}\cdot 2\sin\theta\cos\theta + C$$

$$= \frac{1}{10}\theta - \frac{1}{10}\sin\theta\cos\theta + C$$

Use



$\sec\theta = \frac{5}{y}$, $\sin\theta = \frac{\sqrt{y^2-25}}{5}$

$\cos\theta = \frac{y}{5}$, ~~sec~~ $\theta = \sec^{-1}\left(\frac{5}{y}\right)$

$$= \frac{1}{10}\left(\sec^{-1}(5/y)\right) - \frac{1}{10}\cdot\frac{y}{5}\cdot\frac{\sqrt{y^2-25}}{5} + C$$