5.10 Taylor and Maclaurin Series

Consider the following power series representation:

$$f(x) = \sum_{0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$$

In the last section, we could only write functions into a power series if we could get $f(x)$ into the form $\frac{a}{1-u}$ by differentiating or integrating. A natural question is, "Is there a formula for c_n based on $f(x)$?" One that works for any function $f(x)$.

We do this by finding a pattern while taking derivatives of $f(x)$.

1. $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \ldots$

$$f(a) = c_0$$

2. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots$

$$f'(a) = c_1$$

3. $f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots$

$$f''(a) = 2c_2$$

$$c_2 = \frac{f''(a)}{2}$$

4. $f'''(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \ldots$

$$f'''(a) = 3\cdot 2c_3$$

$$c_3 = \frac{f'''(a)}{3\cdot 2}$$

5. $f^4(x) = 4\cdot 3\cdot 2c_4 + \ldots$

$$f^4(a) = 4\cdot 3\cdot 2c_4$$

$$c_4 = \frac{f^4(a)}{4 \cdot 3 \cdot 2}$$

It appears that if $f(x)$ has a power series representation, then

$$c_n = \frac{f^n(a)}{n!}$$

The next theorem will pretty much state the same thing, but a bit more formally.

Theorem 5.8. *If f has a power series expansion at $x = a$, that is, if*

$$f(x) = \sum_0^{\infty} c_n (x-a)^n$$

with $|x - a| < R$, then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Plug everything back in and we get:

$$f(x) = \sum_0^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{n!}(x-a)^4 + \dots$$

This is called the **Taylor Series** of f centered at $x = a$.

When $a = 0$, we get the **Maclaurin Series**.

$$f(x) = \sum_0^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Let's see how this works. Let's find the Maclaurin Series of $f(x) = \dfrac{1}{1-x}$. Recall we that $\dfrac{1}{1-x} = \sum_{0}^{\infty} x^n$.

1. Set up a table that organizes the derivatives and c_ns.

$$\begin{aligned}
f(x) &= \frac{1}{1-x} &\rightarrow\quad c_0 &= \frac{f(0)}{0!} = 1\\
f'(x) &= \frac{1}{(1-x)^2} &\rightarrow\quad c_1 &= \frac{f'(0)}{1!} = 1\\
f''(x) &= \frac{2}{(1-x)^3} &\rightarrow\quad c_2 &= \frac{f''(0)}{2!} = 1\\
f'''(x) &= \frac{6}{(1-x)^4} &\rightarrow\quad c_3 &= \frac{f'''(0)}{3!} = 1\\
f^4(x) &= \frac{24}{(1-x)^5} &\rightarrow\quad c_4 &= \frac{f^4(0)}{4!} = 1\\
&\vdots
\end{aligned}$$

2. Next, we try to come up with a formula for c_n. I think it's safe to say

$$c_n = 1$$

3. Write out the terms of the series

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$f(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

4. Put it back into series notation

$$f(x) = \sum_{0}^{\infty} x^n$$

Example 5.67. Find the Maclaurin Series of $f(x) = e^x$ and its radius of convergence.

1. Set up a table that organizes the derivatives and c_ns.

$$\begin{aligned}
f(x) = e^x &\rightarrow c_0 = \frac{f(0)}{0!} = \frac{1}{0!} \\
f'(x) = e^x &\rightarrow c_1 = \frac{f'(0)}{1!} = \frac{1}{1!} \\
f''(x) = e^x &\rightarrow c_2 = \frac{f''(0)}{2!} = \frac{1}{2!} \\
f'''(x) = e^x &\rightarrow c_3 = \frac{f'''(0)}{3!} = \frac{1}{3!} \\
f^4(x) = e^x &\rightarrow c_4 = \frac{f^4(0)}{4!} = \frac{1}{4!} \\
&\vdots
\end{aligned}$$

2. Next, we try to come up with a formula for c_n. It appears

$$c_n = \frac{1}{n!}$$

3. Write out the terms of the series

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

$$f(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

4. Put it back into series notation

$$f(x) = \sum_{0}^{\infty} \frac{1}{n!} x^n$$

5. Next, let's find the radius of convergence. We do this by using the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{x!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

for all x.

Since $L = 0 < 1$, the Ratio Test concludes the radius of convergence is $R = \infty$ and the interval of convergence is $(-\infty, \infty)$.

The following statement may sound a bit strange, but here it goes. If e^x has a power series representation, the above work shows us it must be $e^x = \sum_0^\infty \frac{x^n}{n!}$. BUT... we still have to answer this question, "Does e^x actually have a power series representation?"

Let's consider the n-th degree Taylor Polynomial

$$T_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i$$

For example, for $e^x = \sum_0^\infty \frac{1}{n!}x^n$

$$T_1(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i = 1 + x$$

$$T_2(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i = 1 + x + \frac{1}{2!}x^2$$

$$T_3(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

$$T_4(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}(x-a)^i = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4$$

So $T_n(x)$ is a polynomial approxmiation to $f(x)$. If we were to let $n \to \infty$, then

$$f(x) = \lim_{n\to\infty} T_n(x) = \sum_0^\infty \frac{1}{n!}x^n$$

Just like when we did partial sums for a series, if we only go out n terms of the series, we have some remainder.

$$f(x) = T_n(x) + R_n(x)$$

,

where $R_n(x)$ is called the **Remainder** of the Taylor Series. If we can show

$$\lim_{n\to\infty} R_n(x) = 0$$

then this proves $\sum_{0}^{\infty} \frac{1}{n!}x^n$ is really the power series representation for $f(x) = e^x$. We prove this by the following theorem,

Theorem 5.9. *If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the n-th degree Taylor polynomial of f at $x = a$ and*

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor Series on the interval $|x-a| < R$.

In order to show $\lim_{n\to\infty} R_n(x) = 0$, we use the following theorem

Theorem 5.10 (Taylors Inequality)**.** *If $|f^{n+1}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor Series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| < d$.

Our goal is to find M, take the limit of $\frac{M}{(n+1)!}|x-a|^{n+1}$ as $n \to \infty$ and hope it goes to 0. This will force $R_n(x) \to 0$.

1. Think of M as the largest value $f^{n+1}(x)$ can take on over its interval $|x-0|<d$. For our function $f^{n+1}(x)=e^x$, it will take on its largest value at $x=d$.

$$e^x \le e^d \text{ on the interval } -d<x<d$$

2. So let $M=e^d$. Note, $a=0$

$$|R_n(x)| \le \frac{e^d}{(n+1)!}|x|^{n+1}$$

and

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = 0 \text{ for all } x$$

3. Therefore, $R_n(x)\to 0$. This finally shows that

$$e^x = \sum_0^\infty \frac{x^n}{n!}$$

4. Also, if $x=1$, we get

$$e^1 = e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

.

Example 5.68. Suppose we wanted to find the Taylor Series for $f(x)=e^x$ at $a=2$.

1. First, organize a table to find c_n

$$\begin{aligned}
f(x) = e^x &\rightarrow c_0 = \frac{f(2)}{0!} = \frac{e^2}{0!} \\
f'(x) = e^x &\rightarrow c_1 = \frac{f'(2)}{1!} = \frac{e^2}{1!} \\
f''(x) = e^x &\rightarrow c_2 = \frac{f''(2)}{2!} = \frac{e^2}{2!} \\
f'''(x) = e^x &\rightarrow c_3 = \frac{f'''(2)}{3!} = \frac{e^2}{3!} \\
f^4(x) = e^x &\rightarrow c_4 = \frac{f^4(2)}{4!} = \frac{e^2}{4!} \\
&\vdots
\end{aligned}$$

2. Based on the above table, $c_n = \dfrac{e^2}{n!}$.

3. So $f(x) = e^x = \displaystyle\sum_0^{\infty} c_n(x-2)^n = \sum_0^{\infty} \frac{e^2}{n!}$

Example 5.69. Let's find the Maclaurin series for $\cos(x)$

1. First, organize a table to find c_n.

$$\begin{aligned}
f(x) = \cos(x) &\rightarrow \frac{f(0)}{0!} = 1 \\
f'(x) = -\sin(x) &\rightarrow \frac{f'(0)}{1!} = 0 \\
f''(x) = -\cos(x) &\rightarrow \frac{f''(0)}{2!} = -\frac{1}{2!} \\
f'''(x) = \sin(x) &\rightarrow \frac{f'''(0)}{3!} = 0 \\
f^4(x) = \cos(x) &\rightarrow \frac{f^4(0)}{4!} = \frac{1}{4!} \\
f^5(x) = -\sin(x) &\rightarrow \frac{f^5(0)}{5!} = 0 \\
f^6(x) = -\cos(x) &\rightarrow \frac{f^6(0)}{6!} = -\frac{1}{6!} \\
f^7(x) = \sin(x) &\rightarrow \frac{f^7(0)}{7!} = 0 \\
f^8(x) = \cos(x) &\rightarrow \frac{f^8(0)}{8!} = \frac{1}{8!} \\
&\vdots
\end{aligned}$$

2. Well it might be hard to determine a formula for c_n from here. Let's write out the terms of the series and see if that helps.

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

- . . .

It looks like the degree and the factorial go up by 2. So we can write the series like this

$$\cos(x) = \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

3. To check that this is truely the power series representation, let's use Taylor's Inequality to show $R_n(x) \to 0$.

   (a) Note that the derivatives alternate between $\cos(x)$ and $\sin(x)$. So...

$$|f^{n+1}| \leq 1 \text{ for all } x$$

(b) This means we set $M = 1$.

(c) Now onto Taylor's Inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

(d) As $n \to \infty$,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

(e) Therefore, $R_n(x) \to 0$, and we have

$$\cos(x) = \sum_{0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

Example 5.70. Use the previous example to find the Maclaurin Serires for $\sin(x)$.

Since $\sin(x)$ is the anti-derivative of $\cos(x)$, we just integrate the Maclaurin Series for $\cos(x)$.

$$\begin{aligned}
\sin(x) &= \int \cos(x)\, dx \\
&= \int \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}\, dx \\
&= \sum_{0}^{\infty} \int \frac{(-1)^n}{(2n)!}x^{2n}\, dx \\
&= \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Example 5.71. How about $f(x) = x^3 \sin(x)$?

It's actually very simple. We already know the power series representation for $\sin(x)$. Now we just multiply it by x^3.

$$x^3 \sin(x) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)!}$$

Example 5.72. How about $f(x) = \cos(\pi x^2)$

$$\cos(\pi x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{4n}}{(2n)!}$$

5.10.1 Important Maclaurin Series to Memorize

$$\frac{1}{1-x} = \sum_0^\infty x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$e^x = \sum_0^\infty \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = \sum_0^\infty \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\tan^{-1}(x) = \sum_0^\infty \frac{x^{2n+1}}{2n+1} = x - \frac{x}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Example 5.73. Estimate $\displaystyle\int_0^1 e^{-x^2}\, dx$

1. Find the Maclaurin Series for e^{-x^2}

$$e^{-x^2} = \sum_0^\infty \frac{(-x^2)^n}{n!} = \sum_0^\infty (-1)^n \frac{x^{2n}}{n!}$$

2. Now we integrate the series

$$\int_0^1 e^{-x^2}\,dx = \sum_0^\infty (-1)^n \frac{2n+1}{(2n+1)n!}\Bigg|_0^1$$

3. Let's write out the first few terms of this series

$$\int_0^1 e^{-x^2}\,dx = x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \dots\Bigg|_0^1$$

4. Note that when we plug in $x = 0$, we get 0. So we just have to plug in $x = 1$.

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \dots$$

5. Unless you're given an estimate error, just add up the first few terms.

$$1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0.7575$$

6. Since this is an alternating series, we can estimate the error. It must be less than than the next term of the series.

$$\text{Error } < \frac{1}{11\cdot 5!} < 0.000756$$

So our estimate is off by at most 0.000756.

Example 5.74. Find $T_3(x)$ when $f(x) = \tan^{-1}(x)$ at $a = 1$.

1. Let's start with a table to determine c_n.

$f^n(x)$	$f^n(1)$	$c_n = \dfrac{f^n(1)}{n!}$
$f(x) = \tan^{-1}(x)$	$f(1) = \pi/4$	$c_0 = \dfrac{\pi/4}{0!} = \pi/4$
$f'(x) = \dfrac{1}{1+x^2}$	$f'(1) = \dfrac{1}{2}$	$c_1 = \dfrac{1/2}{1!} = \dfrac{1}{2}$
$f''(x) = -\dfrac{2x}{(1+x^2)^2}$	$f''(1) = -\dfrac{2}{4}$	$c_2 = \dfrac{-2/4}{2!} = -\dfrac{1}{4}$
$f'''(x) = \dfrac{6x^2-2}{(x^2+1)^3}$	$f'''(1) = \dfrac{4}{8}$	$c_3 = \dfrac{4/8}{3!} = \dfrac{1}{12}$

2. Use $T_3(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3$

3. $T_3(x) = \dfrac{\pi}{4} + \dfrac{1}{2}(x-1) - \dfrac{1}{4}(x-1)^2 + \dfrac{1}{12}(x-1)^3$

Example 5.75. Let $f(x) = e^{x^2}$, $a = 0$, on the interval $[0, 0.1]$.

1. Find $T_3(x)$

$f^n(x)$	$f^n(0)$	$c_n = \dfrac{f^n(0)}{n!}$
$f(x) = e^{x^2}$	$f(0) = 1$	$c_0 = \dfrac{1}{0!} = 1$
$f'(x) = e^{x^2} \cdot 2x$	$f'(0) = 0$	$c_1 = \dfrac{0}{1!} = 0$
$f''(x) = e^{x^2} \cdot 2 + 2xe^{x^2} \cdot 2x$	$f''(0) = 2$	$c_2 = \dfrac{2}{2!} = 1$
$f'''(x) = e^{x^2}(12x + 8x^3)$	$f'''(0) = 0$	$c_3 = \dfrac{0}{3!} = 0$

2. Write out $T_3(x) = c_0 + c_1x + c_2x^2 + c_3x^3$

$$T_3(x) = 1 + 0x + 1x^2 + 0x^3 = 1 + x^2$$

3. Find the maximum error over the interval $[0, 0.1]$.

$$|R_3(x)| \leq \frac{M}{4!}|x|^4$$

   (a) Let's bound $|x|^4$

$$\begin{aligned} 0 \leq x \leq 0.1 &\rightarrow |x - 0| \leq 0.1 \\ &\rightarrow |x| \leq 0.1 \\ &\rightarrow |x|^4 \leq 0.1^4 \end{aligned}$$

   (b) Let's bound M

$$\begin{aligned} |f^{(4)}(x)| &= |e^{x^2}(12 + 48x^2 + 16x^4)| \\ &\leq |f^{(4)}(0.1)| \\ &\leq e^{.1^2}(12 + .48 + 0.0016) \end{aligned}$$

4. Find the error $|R_3(x)|$

$$|R_3(x)| \leq \frac{e^{.1^2}(12 + .48 + 0.0016)}{4!}(0.1)^4 = 0.00006$$

5. Conclusion: Over the interval $[0, 0.1]$, $f(x) = e^{x^2}$ and $T_3(x) = 1 + x^2$ differ by at most 0.00006.