5 Sequences and Series

5.1 Sequences

A **sequence** is a list of numbers in a definite order.

- a_1 is the first term
- a_2 is the second term
- a_n is the n-th term

The sequence $\{a_1, a_2, a_3, ..., a_n, ...,\}$ is a sequence and we denote it by

$$\{a_n\} \text{ or } \{a_n\}|_{n=1}^{\infty}$$

Examples of Sequences

1) $\left\{\dfrac{n^2}{n^2+1}\right\}$ $\quad a_n = \dfrac{n^2}{n^2+1}$ $\quad \left\{\dfrac{1}{2}, \dfrac{4}{5}, \dfrac{9}{10}, \dfrac{16}{17}, ..., \dfrac{n^2}{n^2+1}, ...\right\}$

2) $\left\{\left(-\dfrac{1}{2}\right)^n (n+3)\right\}$ $\quad a_n = \left(-\dfrac{1}{2}\right)^n (n+3)$ $\quad \left\{\dfrac{4}{-2}, \dfrac{5}{4}, \dfrac{6}{-8}, \dfrac{7}{16}, \dfrac{8}{-32}, ...\right\}$

3) $\left\{\cos\left(\dfrac{2\pi}{n}\right)\right\}$ $\quad a_n = \cos\left(\dfrac{2\pi}{n}\right)$ $\quad \left\{1, -1, -\dfrac{1}{2}, 0, \cos(2\pi/5), \dfrac{1}{2}, \cos(2\pi/7), \dfrac{\sqrt{2}}{2}, ...\right\}$

Some sequences aren't defined so clearly. Here's an example of a sequence that's defined recursively. Recursively means the next term in the sequence is determined by previous terms. The next sequence is probably one of the more famousj sequences.

Example: Fibonacci Sequence

$\{f_n\}$ defined recursively

1. $f_1 = 1$
2. $f_2 = 1$
3. $f_3 = 2$

4. $f_4 = 3$

5. $f_5 = 5$

6. $f_6 = 8$

7. $f_n = f_{n-1} + f_{n-2}$

The next term in the Fibonacci Sequence is the sum of the previous two terms.

Definition 5.1 (Limit of a Sequence)**.** A sequence $\{a_n\}$ has the limit L and we write $\lim_{n\to\infty} a_n = L$ or simpliy $a_n \to L$ as $n \to \infty$ if we can make the terms of a_N as close to L as we make n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say it converges. Otherwise, we say it diverges.

Definition 5.2 (The Formal Definition of Convergence)**.** We will not really use this definition. It is, however, good to know.

A sequence $\{a_n\}$ has a limit L if for every $\epsilon > 0$, there is a corresponding integer N such that if $n > N$, then $|a_n - L| < \epsilon$

This basically says, If you believe the limit of this sequeence is L, then find me the Nth term such that every term after a_N is within ϵ of L.

Example 5.1. Let $a_n = \dfrac{1}{n}$. I claim the limit is 0, i.e., $a_n \to 0$ as $n \to \infty$.

Let $\epsilon = .1$. My job is to find the term where a_n is finally within 0.1 of $L = 0$.

$$a_n = \frac{1}{N} = 0.1$$

$$\frac{1}{0.1} = N$$

$$N = 10$$

So after $N = 10$, every term of a_n is within 0.1 of $L = 0$.

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}, \frac{1}{10}, \underline{\frac{1}{11}}, \underline{\frac{1}{12}}, \underline{\frac{1}{13}}, \underline{\frac{1}{14}}, \ldots, \right\}$$

Do you see how after the 10-th term, a_n is within 0.1 of $L = 0$?

Definition 5.3 (Divergence)**.** $\lim\limits_{n\to\infty} = \infty$, means that for every positive number M, there is an integer N such that if $n > N$, then $a_n > M$.

This definition basically says, "'If you claim the limit is ∞, then I'll give you a very large number (like 100,000,000). You need to tell me the term in a_n where $a_n > 100,000,000$. By doing this for any extremely large number, you essentiall prove $\lim_{n\to\infty} a_n = \infty$

Let's talk **Limit Laws**

If a_n and b_n are convergent sequences and c is some constant:

1. $\lim\limits_{n\to\infty} a_n \pm b_n = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} c \cdot a_n = c \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} a_n \cdot b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
4. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim\limits_{n\to\infty} \neq 0$
5. $\lim\limits_{n\to\infty} (a_n)^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$

Before we do some examples, we have two very useful theorems.

5.1.1 Sequence Theorems

Theorem 5.1 (The Squeeze Theorem)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq N$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then*

$$\lim_{n\to\infty} b_n = L$$

Another (not named) theorem

If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$

Ok, ok.. now some Examples

Example 5.2. Write out the first five terms of $a_n = \dfrac{2n}{n^2+1}$

$$\left\{\frac{2}{2}, \frac{4}{5}, \frac{6}{10}, \frac{8}{17}, \frac{10}{26}, \ldots\right\}$$

It appears the sequence converges to 0, $a_n \to 0$.

Example 5.3. Write out the first five terms of $a_n = \dfrac{(-1)^n}{(n+1)!}$

Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

$$\left\{\frac{-1}{2}, \frac{1}{6}, \frac{-1}{25}, \frac{1}{120}, \frac{-1}{720}, \ldots\right\}$$

This is called an alternating sequence since the signs alternate between -1 and 1. It also appears the sequence converges to 0.

Example 5.4. Write out the first five terms for the sequence that's defined as $a_1 = 6, a_{n+1} = \frac{a_n}{n}$

Note that $n = 1$ doesn't mean a_1, $n = 2$ doesn't mean a_2.

- $a_1 = 1$
- $a_2 = a_{1+1} = \dfrac{a_1}{1} = \dfrac{1}{1} = 1$
- $a_3 = a_{2+1} = \dfrac{a_2}{2} = \dfrac{1}{2}$
- $a_4 = a_{3+1} = \dfrac{a_3}{3} = \dfrac{1/2}{3} = \dfrac{1}{6}$
- $a_5 = a_{4+1} = \dfrac{a_4}{4} = \dfrac{1}{24}$

Example 5.5. Find $\displaystyle\lim_{n\to\infty} \frac{n}{n+1}$

Most of these can be done using methods we already know.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n}{n+1} &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}$$

Just to verify this, let's look at the first few terms and see if it appears the sequence approaches 1.

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{7}{8}, ..., \frac{99}{100}, ..., \frac{99999}{100000}, ... \right\}$$

Example 5.6. Find $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n}$

Ooooh, I think we have a rule that helps with this. What was it called...L'Hospital's Rule!

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln n}{n} &\overset{LH}{\to} \lim_{n\to\infty}\frac{1/n}{1}\\ &= 0\end{aligned}$$

Example 5.7. Is $a_n = (-2)^{n-1}$ convergent or divergent?

$$\{1, -2, 4, -8, 16, -32, 64, -128, 256, ...\}$$

It appears a_n is diverging. The numbers keep getting larger and larger. They are also alternating. Some of the terms appear to approach ∞ while the others $-\infty$. This really just adds more to why this sequence diverges.

Example 5.8. Does $a_n = \dfrac{(-1)^n}{n}$ converge?

Yes. If you look at the sequence

$$\left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, ..., \frac{1}{100}, \frac{-1}{101}, ...\right\}$$

It appears to converge to 0. So how can we prove that? Let's use our **un-named theorem** from a couple of pages ago.

$$|a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

We know $\dfrac{1}{n} \to 0$ as $n \to \infty$. Therefore, since $|a_n| = 0$, a_n must converge to 0.

Example 5.9. Let $a_n = \cos(\pi/n)$

$$\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1$$

Example 5.10. Where is $a_n = r^n$, r is a constant, convergent?

Through a little work, which I'll show you in class, we have

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}$$

This is called a **GEOMETRIC SEQUENCE**.

Example 5.11. Some quick examples

1. $\displaystyle\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$ since $-1 < \dfrac{2}{3} < 1$

2. $\displaystyle\lim_{n\to\infty} \frac{5^n}{3^n} = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n$ diverges because $\dfrac{5}{3} > 1$

3. $\displaystyle\lim_{n\to\infty} 1^n = 1$ since $r = 1$

4. $\displaystyle\lim_{n\to\infty} \left(\frac{-7}{8}\right)^n - 5 = 0 - 5$ converges to -5

Example 5.12. Does $a_n = \dfrac{n^2 - 4^n}{7^n}$ converge?

Let's check its limit.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^2-4^n}{7^n} &\overset{LH}{\to} \lim_{n\to\infty} \frac{2n-4^n\ln 4}{7^n \ln 7} \\
&\overset{LH}{\to} \lim_{n\to\infty} \frac{2-4^n(\ln 4)^2}{7^n(\ln 7)^2} \\
&\overset{LH}{\to} \lim_{n\to\infty} \frac{-4^n(\ln 4)^3}{7^n(\ln 7)^2} \\
&= \lim_{n\to\infty} -\left(\frac{\ln 4}{\ln 7}\right)^3 \cdot \left(\frac{4}{7}\right)^n \\
&= 0 \text{ since } -1 < \frac{4}{7} < 1
\end{aligned}$$

So yes, $a_n = \dfrac{n^2-4^n}{7^n}$ does converge.

Example 5.13. Does $a_n = \dfrac{5^{n+4}}{3^{n-2}}$ converge?

We need to rewrite this so it's of the form r^n.

$$\begin{aligned}
\lim_{n\to\infty} \frac{5^{n+4}}{3^{n-2}} &= \lim_{n\to\infty} \frac{5^4\cdot 5^n}{3^{-2}\cdot 3^n} \\
&= \lim_{n\to\infty} 5^4\cdot 3^2 \cdot \left(\frac{5}{3}\right)^n \\
&= \infty \text{ since } \frac{5}{3} > 1
\end{aligned}$$

Example 5.14. Does $a_n = \dfrac{\cos^2(n)}{2^n}$ converge?

We know

$$-1 \le \cos^2(n) \le 1$$

Divide all sides by 2^n

$$-\frac{1}{2^n} \le \frac{\cos^2(x)}{2^n} \le \frac{1}{2^n}$$

Since $\lim_{n\to\infty} -\dfrac{1}{2^n} = \lim_{n\to\infty} \dfrac{1}{2^n} = 0$, then by the **Squeeze Theorem**, $\lim_{n\to\infty} \dfrac{\cos^2(n)}{2^n} = 0$

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One of the biggest skills you can develop during the sequence and series section is to be able to look at a sequence and make an educated (and hopefully) correct guess on whether the sequence converges.

Here's a list of terms you'll come across and how they rank among each other when n is very large.

$$c < \ln n < \text{any polynomial} < \text{exponential functions (base} > 1) < n! < n^n$$

An example would be

$$5 < 5\ln n^{200} + 515n^3 < 3^n < n! < n^n$$

Example 5.15. Does $a_n = \dfrac{n!}{n^n}$ converge?

Based on what I wrote above, this should converge to 0. But let's go ahead and prove it.

Let's take a look at some of the terms.

- $a_1 = \dfrac{1}{1}$
- $a_2 = \dfrac{2 \cdot 1}{2 \cdot 2}$
- $a_3 = \dfrac{3 \cdot 2 \cdot 1}{3 \cdot 3 \cdot 3}$

Following the pattern, we can write

$$a_n = \frac{n(n-1)(n-2)(n-3)\cdot ... \cdot 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdot n \cdot ... \cdot n \cdot n}$$

Each term on the numerator pairs with an n on the denominator.

$$a_n = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot ... \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n}$$

Rewrite it as

$$a_n = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot ... \cdot \frac{n-2}{n} \cdot \frac{n-1}{n} \cdot \frac{n}{n}$$

$$a_n = \frac{1}{n} \left(\frac{2}{n} \cdot \frac{3}{n} \cdot ... \cdot \frac{n-2}{n} \cdot \frac{n-1}{n} \cdot \frac{n}{n} \right)$$

Since every product in the parentheses is ≤ 1, if you get rid of them, we have

$$0 < a_n \leq \frac{1}{n}$$

By the **Squeeze Theorem**, since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

We're almost done. Let's just go over a couple more definitions before we move on.

Definition 5.4. A sequence is increasing if $a_n < a_{n+1}$ for $n \geq 1$.

A sequence is decreasing if $a_n > a_{n+1}$ for $n \geq 1$.

A sequence is **monotonic** if it is always increasing or always decreasing.

Example 5.16. $a_n = \dfrac{3}{n+5}$.

If you write out the first few terms, you'll see a_n is decreasing to 0.

$$\left\{ \frac{3}{6}, \frac{3}{7}, \frac{3}{8}, \frac{3}{9}, \frac{3}{10}, ..., \right\}$$

So a_n a decreasing monotonic sequence.

$a_n = 1 - \dfrac{1}{n}$

The first few terms are

$$\left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots\right\}$$

So a_n converges to 1. Since it's always increasing, it's monotonic.

$a_n = \dfrac{(-1)^n}{n}$

The first few terms are

$$\left\{-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \frac{-1}{7}, \ldots\right\}$$

We can see a_n converges to 0. But it's not always increasing and not always decreasing. This is an example of a non-monotonic sequence.

Definition 5.5. A sequence is bounded above if there is a number M such that $a_n \leq M$ for $n \geq 1$.

A sequence is bounded below if there is a number m such that $a_n \geq m$ for $n \geq 1$.

A sequence is called **bounded** if it is bounded from above and below.

Example 5.17.

1. $a_n = 1 - \dfrac{1}{n}$ is bounded below by 0 and above by 1.

2. $a_n = \dfrac{(-1)^n}{n}$ is bounded below by -1, and above 1.

3. $a_n = n$ is bounde below by 1, but it is not bounded above since $\lim\limits_{n\to\infty} n = \infty$