Math 230

Calculus II

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Review

1 Review

Welcome to Calculus II. And congrats on passing Calculus I. As I mentioned in my calculus I notes, one of the hardest parts of calculus is algebra. You still need to know how to factor and simplify nasty expressions. Now that you're in Calculus II, you also need to have mastered all of Calculus I's differentiation formulas and the basic integration techniques. We will go through some u-substitution problems in a little bit. You must be able to do these as these will be the easiest integration problems all semester.

Calculus II is made up of two parts. The first part covers many integration techniques and some applications. The second part covers sequences and series. It's a very interesting part of the course. I hope you're not too exhausted by the time we get there.

Please do not use these notes as a substitute for coming to class. Even though these are my lecture notes, you miss out on my fantastic explanations.

These notes are always being updated. If you find any mistakes, typos, etc., please let me know by sending me an email at bveitch@niu.edu

With all that out of the way, let's get started!

Differentiation

Let's start by summarizing them many main points from last semester. Recall from Calculus I the following differentiation formulas,

Formula 1: Differentation Formulas

1. Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
2. Product Rule: If $y = f(x) \cdot g(x)$, then $y' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$
3. Quotient Rule: If $y = \dfrac{f(x)}{g(x)}$, then $y' = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
4. Chain Rule: If $y = f(g(x))$, then $y' = f'(g(x)) \cdot g'(x)$.

You can find differentiation examples in my Calculus I notes.

Theorem 1: Fundamental Theorem of Calculus and Common Anti-derivatives

Part 1: If f is continuous on [a,b], then $F(x) = \int_a^x f(t)dt$ is continuous on [a,b] and differentiable on (a,b) and

$$F'(x) = f(x)$$

Another way of writing this is

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Basically it says integration and differentiation are inverses of each other. If you integrate $f(x)$ and then differentiate it, you get back $f(x)$.

Part 2: If f is continuous at every point in [a,b] and F is any anti-derivative of f on [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Definition 1: Common Integrals

1. $\int K \; dx = Kx + C$
2. $\int x^n \; dx = \frac{x^{n+1}}{n+1} + C$
3. $\int \sin(x) \; dx = -\cos(x) + C$
4. $\int \cos(x) \; dx = \sin(x) + C$
5. $\int \sec(x)\tan(x) \; dx = \sec(x) + C$
6. $\int \sec^2(x) \; dx = \tan(x) + C$

Definition 2: u-substitution

If $g(x)$ is differentiable and f is continuous, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

One way to recognize a substitution is much like recognizing a chain rule. You need to identify an inside function and an outside function. In this case, $f(x)$ is the outside function and $g(x)$ is the inside function. If you also see the derivative of the inside function $g'(x)$ somewhere in the integral, then it's a good chance it's a u-substitution.

Before moving on to the new stuff, let's do some examples of integration using substitution.

Example 1

$\int \sqrt{2x+1}\,dx$

1. Let $u = 2x+1$
2. $du = 2\,dx \to \frac{du}{2} = dx$.
3. Now make the substitution.

$$\begin{aligned}\int \sqrt{2x+1} &= \int \sqrt{u}\cdot\frac{du}{2}\\ &= \frac{1}{2}\int u^{1/2}\,du\\ &= \frac{1}{2}u^{3/2}\cdot\frac{2}{3}\\ &= \frac{1}{3}u^{3/2}\end{aligned}$$

4. Substitute back

$$\frac{1}{3}(2x+1)^{3/2} + C$$

Don't forget the $+C$. All indefinite integrals will have them.

Example 2

$$\int \frac{3}{(2-x)^2}\,dx$$

1. Let $u = 2 - x$
2. $du = -1\,dx \to -1\,du = dx$
3. Now make the substitution.

$$\begin{aligned}\int \frac{3}{(2-x)^2}\,dx &= \int \frac{3}{u^2}\,(-du)\\ &= \int 3u^{-2}\,du\\ &= \frac{-3}{-1}u^{-1} + C\\ &= 3u^{-1} + C\\ &= \frac{3}{u} + C\end{aligned}$$

4. Substitute back

$$\frac{3}{(2-x)} + C$$

Example 3

$$\int_0^1 t^3(1+t^4)^3\,dt$$

1. Let $u = 1 + t^4$
2. $du = 4t^3\,dt \to \frac{du}{4} = t^3\,dt$

3. Since this is a definite integral (it has bounds), we must find our new bounds in terms of u.

$$\text{If } x = 0 \rightarrow u = 1 + (0)^4 = 1$$

$$\text{If } x = 1 \rightarrow u = 1 + (1)^4 = 2$$

4. Now make the substitution and evaluate

$$\begin{aligned}\int_0^1 t^3(1+t^4)^3\,dt &= \int_1^2 (u)^3 \frac{du}{4} \\ &= \frac{1}{4}\int_1^2 u^3\,du \\ &= \left.\frac{u^4}{16}\right|_1^2 \\ &= 1 - \frac{1}{16} \\ &= \frac{15}{16}\end{aligned}$$

Example 4

$$\int \frac{1}{\theta^2}\sin\left(\frac{1}{\theta}\right)\cos\left(\frac{1}{\theta}\right)\,d\theta$$

This one is a bit trickier. One of the first things you look for is what's in parenthesis. A lot of time we let u equal what's in there, well...because that's usually the case. But not this time!

1. Let $u = \sin\left(\frac{1}{\theta}\right)$

2. $du = \cos\left(\frac{1}{\theta}\right) \cdot -\frac{1}{\theta^2}\,d\theta \rightarrow -du = \cos\left(\frac{1}{\theta}\right) \cdot \frac{1}{\theta^2}\,d\theta$

3. Make the substitution

$$\begin{aligned}\int \frac{1}{\theta^2}\sin\left(\frac{1}{\theta}\right)\cos\left(\frac{1}{\theta}\right)\,d\theta &= \int u\,(-du)\\ &= \int -u\,du\\ &= -\frac{u^2}{2}+C\end{aligned}$$

4. Substitute back

$$-\frac{1}{2}\sin^2\left(\frac{1}{\theta}\right)+C$$

Example 5

$$\int \frac{\sin(\sqrt{t})}{\sqrt{t}}\,dt$$

1. Let $u=\sqrt{t}$
2. $du=\frac{1}{2\sqrt{t}}\,dt \to 2\,du=\frac{1}{\sqrt{t}}\,dt$
3. Make the substitution

$$\begin{aligned}\int \frac{\sin(\sqrt{t})}{\sqrt{t}}\,dt &= \int \sin(u)\,(2du)\\ &= 2\int \sin(u)\,du\\ &= -2\cos(u)+C\end{aligned}$$

4. Substitute back

$$-2\cos(\sqrt{t})+C$$

Example 6

$$\int_0^{\pi/4} \tan(x)\sec^2(x)\ dx$$

1. Let $u = \tan(x)$
2. $du = \sec^2(x)\ dx$
3. Since this is a definite integral, we need to adjust the bounds.

$$\text{If } x = 0 \to u = \tan(0) = 0$$

$$\text{If } x = \pi/4 \to u = \tan(\pi/4) = 1$$

4. Make the substitution

$$\begin{aligned} \int_0^{\pi/4} \tan(x)\sec^2(x)\ dx &= \int_0^1 u\ du \\ &= \frac{1}{2}u^2\Big|_0^1 \\ &= \frac{1}{2} - 0 \\ &= \frac{1}{2} \end{aligned}$$