5.9 Representations of Functions as a Power Series

Example 5.58. The following geometric series

$$\sum x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

will converge when $-1 < x < 1$. We also know that a geometric series with radius x will converge to

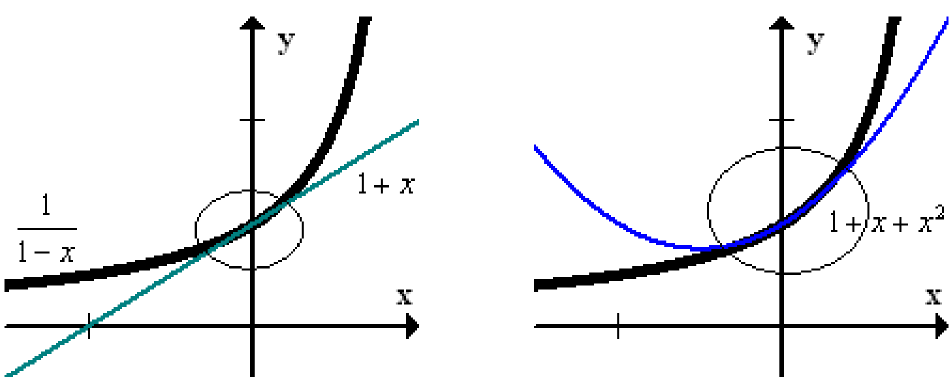
$$\sum x^n = \frac{1}{1-x}$$

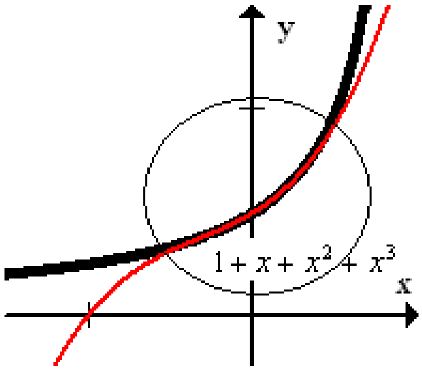
So what did we just do? We found a power series representation for the function $f(x) = \dfrac{1}{1-x}$. But, this representation only works for values of x where $-1 < x < 1$.

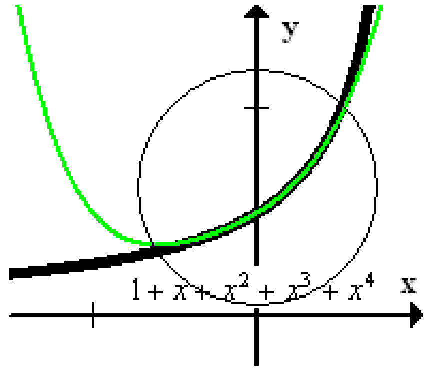
Who cares, right? Wrong! We care. Remember that an infinite series can be approximated by adding up the first N terms. For us, we can approximate $f(x) = \dfrac{1}{1-x}$ by

$$\frac{1}{1-x} \approx 1 + x$$
$$\frac{1}{1-x} \approx 1 + x + x^2$$
$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3$$
$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4$$

All of these approximate our function. However, the more terms we add the better the approximation. Let's take a look at the graph of all these.



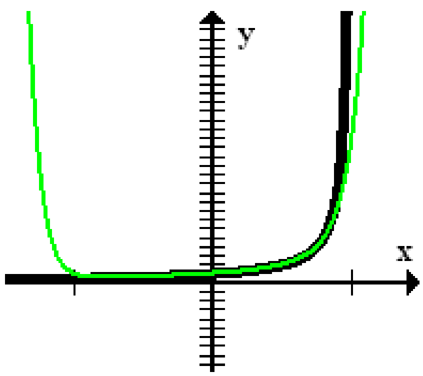




See? More terms equals better approximation. If I use

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14}$$

and zoom out alot, we get this



I want to draw your attention to something. When does the approximation begin to really deviate from $f(x) = \dfrac{1}{1-x}$. It appears to deviate when $|x| \geq 1$. This is exactly when the series $\sum x^n$ diverges. Interesting...

During the last section, we spent a great deal of time determining what values of x will make the following power series $\sum c_n(x-a)^n$ converge.

What we're going to do in this section, is find ways of rewriting some functions so they can be written like a power series.

Example 5.59.

1. Express $\dfrac{4}{1+x^2}$ as a power series.

The trick is rewrite $\dfrac{4}{1+x^2}$ so that it can look something like $\dfrac{a}{1-r} = \sum ar^n$.

$$\frac{4}{1+x^2} = \frac{4}{1-(-x^2)}$$

We can call $a = 4$ and $r = -x^2$. Therefore,

$$\frac{4}{1+x^2} = \sum 4(-x^2)^n = \sum 4(-1)^n x^{2n}$$

Next, let's find the interval ov convergence.

$$L = \lim_{n\to\infty} \left| \frac{4(-1)^{n+1}x^{2(n+1)}}{4(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{(-1)^n x^{2n}} \right| = \lim \left|4x^2\right|$$

which converges when

$$\left|4x^2\right| < 1$$

Solving for x, we get

$$\left|x^2\right| < \frac{1}{4}$$

$$|x| < \frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

2. Express $\dfrac{2}{3-x}$ as a power series.

   Rewrite $\dfrac{2}{3-x}$ as $\dfrac{2}{3(1-x/3)} = \dfrac{2/3}{1-(x/3)}$

   Let $a = \dfrac{2}{3}$ and $r = \dfrac{x}{3}$, we get

$$\frac{2}{3-x} = \sum_{1}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^{n-1}$$

3. Express $\dfrac{x}{2x^2+1}$ as a power series.

   Rewrite $\dfrac{x}{2x^2+1}$ as

$$x \cdot \frac{1}{1+2x^2} = x \cdot \frac{1}{1-(-2x^2)}$$

   If we let $r = -2x^2$, we get

$$\begin{aligned} \frac{x}{2x^2+1} &= x \cdot \frac{1}{1-(-2x^2)} \\ &= x\sum_{0}^{\infty}(-2x^2)^n \\ &= x\sum_{0}^{\infty}(-2)^n x^{2n} \\ &= \sum_{0}^{\infty}(-2)^n x^{2n+1} \end{aligned}$$

   If we write out the first few terms, it would look like

$$\frac{x}{2x^2+1} = x - 2x^3 + 4x^5 - 8x^7 + 16x^9 - 32x^{11} + ...$$

   And this series will converge as long as $|-2x^2| = 2x^2 < 1$.

We've been lucky in that the functions we're working with start off looking similiar to $\dfrac{a}{1-r}$. Now let's see what happens when we differentiate or integrate a series. This could open up a variety of new functions.

5.9.1 Term by Term Differentiation and Integration

Given the function $f(x) = \sum_{0}^{\infty} c_n (x-a)^n$, whose domain is the interval of convergence.

Theorem 5.7. *If a power series* $\sum c_n(x-a)^n$ *has radius* $R > 0$, *then*

$$f(x) = \sum_{0}^{\infty} = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

is differentiable on $(a-R, a+R)$, *and*

1. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots = \sum nc_n(x-a)^{n-1}$

2. $\int f(x)\, dx = c + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + c_3\frac{(x-a)^4}{4} + \dots = c + \sum c_n \frac{(x-a)^{n+1}}{n}$

Example 5.60. Express $\frac{1}{(1-x)^2}$ as a power series.

Is there a function we know of that has a power series representation and can be differentiated or integrated to give us $\frac{1}{(1-x)^2}$.

The answer is yes. Note that

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2}$$

and we know what the power series representation of $\frac{1}{1-x}$. Yay!!!

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} x^n \\ &= \sum_{n=1}^{\infty} nx^{n-1} \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots \end{aligned}$$

You may also approach is like this

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\
&= \frac{d}{dx}\left[1+x+x^2+x^3+x^4+\ldots\right] \\
&= 0+1+2x+3x^2+4x^3+\ldots \\
&= \sum_{n=0}^{\infty} nx^{n-1} \\
&= \sum_{n=1}^{\infty} nx^{n-1}
\end{aligned}$$

Note that the index changed from $n=0$ to $n=1$. We can do this because the first term in that series is 0, and therefore don't need it.

Example 5.61. Find a power series representation for $\ln(1+x)$ and it's radius of convergence.

We know $\int \frac{1}{1+x}\, dx = \ln(1+x)$. So we figure out the power series representation for $\frac{1}{1+x}$ and integrate it to get $\ln(1+x)$.

$$\begin{aligned}
\frac{1}{1+x} &= \frac{1}{1-(-x)} \\
&= \sum_{n=0}^{\infty} (-x)^n \\
&= \sum_{n=0}^{\infty} (-1)^n x^n
\end{aligned}$$

Now let's integrate back

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx \\ &= \sum_{n=0}^{\infty} \int (-1)^n x^n\,dx \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C\end{aligned}$$

Use this to find $\ln(1/2)$.

Since we found the power series representation for $\ln(1+x)$, we need to figure out what x is to get $\ln(1/2)$. Notice that $x = -1/2$.

$$\begin{aligned}\ln(1/2) &= \ln(1+(-1/2)) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(-1/2)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}(1/2)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)(1/2)^{n+1}}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{-1}{(n+1)2^{n+1}}\end{aligned}$$

Example 5.62. Find a power series representation for $f(x) = \ln(1+x)$, $f(x) = x\ln(1+x)$, $f(x) = \ln(x^2+1)$, and $\ln(1-x)$.

1. $f(x) = \ln(1+x)$

   Note that $\dfrac{d}{dx}[\ln(1+x)] = \dfrac{1}{1+x}$ and we know

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{0}^{\infty} (-1)^n x^n$$

Therefore,

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_0 (-1)^n x^n\,dx = \sum_0 (-1)^n \int x^n\,dx = \sum_0 (-1)^n \frac{x^{n+1}}{n+1}$$

2. $f(x) = x\ln(1+x)$.

   Here, we just multiply our series representation for $\ln(1+x)$ by x.

$$x\ln(1+x) = x \cdot \sum_0 (-1)^n \frac{x^{n+1}}{n+1} = \sum_0 \sum (-1)^n \frac{x^{n+2}}{n+1}$$

3. $f(x) = \ln(x^2+1)$

   This series is similiar to $\ln(x+1)$, except we replaced x with x^2. So all we need to do is replace x with x^2 in our power series representation for $\ln(x+1)$ from part (1).

$$f(x) = \ln(x^2+1) = \sum_0 (-1)^n \frac{(x^2)^{n+1}}{n+1} = \sum_0 (-1)^n \frac{x^{2n+2}}{n+1}$$

4. $f(x) = \ln(1-x)$

   Again, recall that we know $\ln(1+x) = \sum_0 (-1)^n \frac{x^{n+1}}{n+1}$. We just need to replace x with $-x$ and we'll have the power series representation for $\ln(1-x)$.

$$\ln(1-x) = \sum_0 (-1)^n \frac{(-x)^{n+1}}{n+1} = \sum_0 (-1)^n \frac{(-1)^{n+1}x^{n+1}}{n+1} = \sum_0 -\frac{x^{n+1}}{n+1}$$

   Note that $(-1)^n \cdot (-1)^{n+1} = -1$

Example 5.63. Find a power series representation for $f(x) = \tan^{-1}(x)$.

Note that $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$. Therefore,

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

Now we just need the power series representation for $\dfrac{1}{1+x^2}$

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= \sum_{n=0}^{\infty}(-x^2)^n \\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n} \end{aligned}$$

Therefore,

$$\begin{aligned} \tan^{-1}(x) &= \int \frac{1}{1+x^2}\,dx \\ &= \int \sum_{n=0} (-1)^n x^{2n}\,dx \\ &= \sum_{n=0} \int (-1)^n x^{2n}\,dx \\ &= \sum_{n=0} (-1)^n \frac{x^{2n+1}}{2n+1} + C \end{aligned}$$

Example 5.64. Find a power series representation for $f(x) = \tan^{-1}(x/3)$.

Like the previous example, we have a power series representation for $f(x) = \tan^{-1}(x)$. So we just replace the x in $\tan^{-1}(x)$ with $(x/3)$.

$$\tan^{-1}(x/3) = \sum_{0} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{0} \frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}$$

Note the radius of convergence is when $\left|\dfrac{x}{3}\right| < 1$.

Example 5.65. Suppose I wanted to evaluate the following integral

$$\int \frac{\ln(1-t^5)}{t}\, dt$$

This isn't a very friendly integral (even for calculus II). Instead, we write our integrand as a power series and integrate that. A power series representation essentially rewrites your function into a polynomial, which if you remember from calc I, is extremely easy to integrate and differentiate.

$$\int \frac{\ln(1-t^5)}{t}\, dt = \int \frac{\sum -\frac{(t^5)^{n+1}}{n+1}}{t}\, dt = \int \frac{\sum -\frac{t^{5n+5}}{n+1}}{t}\, dt = \int -\sum \frac{t^{5n+4}}{n+1}\, dt$$

$$= -\sum \int \frac{t^{5n+4}}{n+1}\, dt = -\sum \frac{t^{5n+5}}{(5n+5)(n+1)}$$

So here are the steps to integrating and differentiating complicated functions using a power series.

1. Figure out how to rewrite your function into a power series. This may take a while (like the previous example).

2. Use the term by term intergration or differentiation technique we utilized in the previous example.

3. Remember that integral and derivative notation can be pulled in or out of an summation.

4. Finally, clean it up by simplifying.

5. If for some reason, it's a definite integral (has bounds), then write out the power series for the first 4 or 5 terms. This way you have a polynomial. Just evaluate the polynomial like you would in any integration problem. Just keep in mind the final answer is only an estimate. The estimate gets better when you use more terms of your series.

Example 5.66.

$$\int_{.1}^{.5} \frac{\ln(1-t^5)}{t}\ d$$

(a) Figure out the power series reprsentation for the integrand (which we did already) and integrate

(b) $-\sum_0 \frac{t^{5n+5}}{(5n+5)(n+1)}\bigg|_{.1}^{.5}$

(c) Write out a few terms of the series

$$-\frac{t^5}{5} - \frac{t^{10}}{20} - \frac{t^{15}}{45} - \frac{t^{20}}{80}\bigg|_{.1}^{.5}$$

$$\left(-\frac{(.5)^5}{5} - \frac{(.5)^{10}}{20} - \frac{(.5)^{15}}{45} - \frac{(.5)^{20}}{80}\right) - \left(-\frac{(.1)^5}{5} - \frac{(.1)^{10}}{20} - \frac{(.1)^{15}}{45} - \frac{(.1)^{20}}{80}\right) = -0.0062975$$