5.8 Power Series

Consider the following series

$$\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

where c_ns are coefficients to x_n.

This is called a power series centered at $a = 0$. A general power series, centered at a, is

$$\sum c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \dots$$

We have already seen some examples of power series. We will focus on two things during these sections. The first is to determine what values of x will allow the series to converge. The second is writing certain functions as a series.

Example 5.55. The following geometric series

$$\sum x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

will converge when $-1 < x < 1$. We also know that a geometric series with radius x will converge to

$$\sum x^n = \frac{1}{1-x}$$

Let's look at some examples of power series and determine what values of x will allow the series to converge.

Example 5.56.

1. For what values of x does $\sum \frac{(x+7)^n}{n^2+1}$ converge?

   We do this by using the Ratio or Root test. In this case, we'll use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{(x+7)^{n+1}}{(n+1)^2+2} \cdot \frac{n^2+1}{(x+7)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+7)}{1} \cdot \frac{n^2+1}{(n+1)^2+1} \right|$$

$$= |x+7| \lim_{n\to\infty} \left| \frac{n^2+1}{(n+1)^2+1} \right| = |x+7| \cdot 1 = |x+7|$$

We know by the Ratio Test, the series will converge absolutely when $L < 1$. For us, $L = |x+7|$. So we solve this inequality to find what values of x will make $L < 1$.

$$|x+7| < 1$$
$$-1 < x+7 < 1$$
$$-8 < x < -6$$

So any value betwten -8 and -6 will make the series converge. But what about $x = -8$ or $x = -6$? Do they make the series converge? You need to check the endpoints of your interval individually.

(a) $x = -8$

$$\sum \frac{(-8+7)^n}{n^2+1} = \sum \frac{(-1)^n}{n^2+1},$$

which converges by the Alternating Series Test.

(b) $x = -6$

$$\sum \frac{(-6+7)^n}{n^2+1} = \sum \frac{1}{n^2+1},$$

which converges by the a Direct Comparison with a p-series $\sum \frac{1}{n^2}$.

Keep in mind that the series may not converge at the endpoints. For example, $\sum \frac{(x+7)^n}{n+1}$, will converge at one of the endpoints but not the other. The endpoints will be the same as the previous example, but will not work at $x = -6$, since

$$\sum \frac{(-6+7)^n}{n+1} = \sum \frac{1}{n+1} \text{ diverges}$$

The lesson is, never assume the series will converge at the endpoints.

2. Find all x-values that allow $\sum_{n=1}^{\infty} n!x^n$

   Let's use the Ratio Test to find L.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} \frac{(n+1)n!x^n x}{n!x^n} = \lim_{n\to\infty} (n+1)x = \infty$$

   Notice that the x is insignificant. As $n \to \infty$, so does L. There is only one x value that will allow this seris to converge and that's $x = 0$. Therefore, the series $\sum_{n=1}^{\infty} n!x^n$ converges only when $x = 0$.

3. Find all x-values that allow $\sum \frac{(-1)^n x^{2n}}{(2n)!}$ to converge.

   Again, we use the Ratio Test to find L.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{2n}x^2}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right| = \lim_{\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| = 0$$

   for all x.

   In this example, it doesn't matter what x is since the limit will approach 0. Therefore, $\sum \frac{(-1)^n x^{2n}}{(2n)!}$ converges for all x or $(-\infty, \infty)$.

There are three possible types of convergence for a power series $\sum c_n(x-a)^n$. The previous three examples demonstrate the three types of convergence. The three types are

1. $\sum c_n(x-a)^n$ converges only when $x = a$, as in example (2)

2. $\sum c_n(x-a)^n$ converges for all x, as in example (3)

3. There is a positive R such that if $|x-a| < R$, $\sum c_n(x-a)^n$ converges. If $|x-a| > R$, then $\sum c_n(x-a)^n$ diveges. This is what happens in example (1) above. The number R is called the **Radius of Convergence**. The intervals of convergence will be centered around $x = a$. You will have to check the endpoints of the interval of convergence separately to determine if the series converges.

Example 5.57. Determine the Radius and Interval of convergence for the following series:

1. $\displaystyle\sum \frac{(-1)^n x^n}{4^n \ln n}$

   As always, use the Ratio Test to find L.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{(-1)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{4^n}{4^{n+1}} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{\ln n}{\ln n + 1} \right| = \left|\frac{x}{4}\right| \lim_{n\to\infty} \frac{\ln n}{\ln n + 1} = \left|\frac{x}{4}\right|$$

   So our series will converge when

$$L = \left|\frac{x}{4}\right| < 1$$

$$|x| < 4$$

   We can see the Radius of convergence R is 4. Solving this inequality, we get $-4 < x < 4$. Now we check the endpoints.

   (a) $x = -4$

$$\sum \frac{(-1)^n(-4)^n}{4^n \ln n} = \sum \frac{4^n}{4^n \ln n} = \sum \frac{1}{\ln n}$$

   which diverges. We can check the divergence by comparing it to the p-series $\sum \frac{1}{n}$.

(b) $x = 4$

$$\sum \frac{(-1)^n(4)^n}{4^n \ln n} = \sum \frac{(-1)^n}{\ln n}$$

which converges by the Alternating Series Test.

Therefore, $\sum \frac{(-1)^n x^n}{4^n \ln n}$ converges on the interval $(-4, 4]$.

2. $\sum \frac{n(5x+2)^n}{3^{n+1}}$

(a) As always, use the Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{(n+1)(5x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(5x+2)^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \cdot \frac{5x+2}{3} \right| = \left| \frac{5x+2}{3} \right|$$

(b) $L = \left| \frac{5x+2}{3} \right|$ will converge when

$$\left| \frac{5x+2}{3} \right| < 1$$

Before solving for x to find the interval of convergence, I want to find the center a and the radius R. We can do it from this point, as long as its in the form $|x - a|$

$$\left| \frac{5}{3} \cdot \left(x + \frac{2}{5} \right) \right| < 1$$

$$\left| x + \frac{2}{5} \right| < \frac{3}{5}$$

From here, we see the radius of convergence is $R = \frac{3}{5}$ with the center $a = -\frac{2}{5}$.

(c) Let's keep solving for x to find the endpoints.

$$-\frac{3}{5} < x + \frac{2}{5} < \frac{3}{5}$$

$$-1 < x < \frac{1}{5}$$

(d) Now we check the endpoints, $x = -1$ and $x = \frac{1}{5}$

i. $x = -1$

$$\sum \frac{n(5(-1)+2)^n}{3^{n+1}} = \sum \frac{n(-3)^n}{3(3)^n} = \sum \frac{(-1)^n n}{3}$$

which diverges by the Test for Divergence.

ii. $x = \frac{1}{5}$

$$\sum \frac{n(5(1/5)+2)^n}{3^{n+1}} = \sum \frac{n(3)^n}{3(3)^n} = \sum \frac{n}{3} \to \infty$$

Therefore, $\sum \frac{n(5x+2)^n}{3^{n+1}}$ converges on the interval $\left(-1, \frac{1}{5}\right)$.