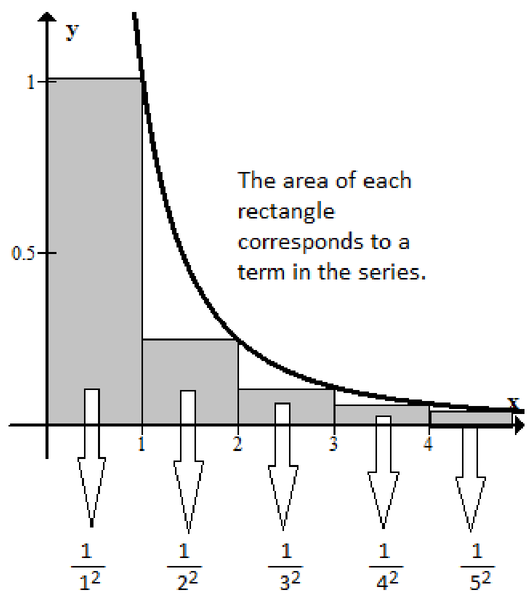
5.3 The Integral Test and Estimates of Sums

The next few sections we learn techniques that help determine if a series converges. In the last section we were able to find the sum of the series. It's difficult to find the sum of a series. We'll spend most of our time now just determining if the series converges.

Consider the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

We have no simple formula for the sequence of partial sums S_n. We're going to compare the terms of a sequence to the graph of $y = \dfrac{1}{x^2}$.



So the area of each of these rectangles is a term in the series. If we were to add up the area of all these rectangles, we would have the sum for the series. Since we're only concerned now with knowing if it converges to a finite number, we're only going to focus on its convergence.

Notice that the area of rectangles is less than the area

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx$$

therefore, we have the following relation

$$\sum_{n=1}^\infty \frac{1}{n^2} < \frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx$$

The natural question at this point is, does $\displaystyle\int_1^\infty \frac{1}{x^2}\, dx$ converge? It's an improper integral and we know how to evaluate those.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\, dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx \\
&= \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t \\
&= \lim_{t\to\infty} -\frac{1}{t} + \frac{1}{1} \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

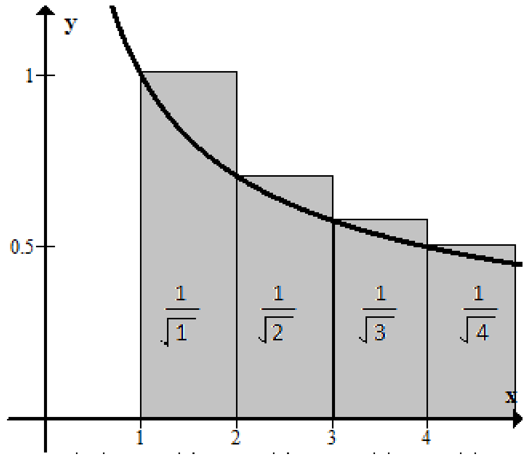
So

$$\sum_{n=1}^\infty \frac{1}{n^2} < \frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\, dx < \frac{1}{1^2} + 1 = 2$$

Since we know our series is made up of all positive terms and we just showed the sum can't be more than 2, we conclude the series converges.

Example 5.31. Let's try to use this idea to determine if $\displaystyle\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ converges.

Let's compare it to the following graph of $y = \dfrac{1}{\sqrt{x}}$.



Notice how the area of the each rectangle is larger than the area under $y = \dfrac{1}{\sqrt{x}}$. This means

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$$

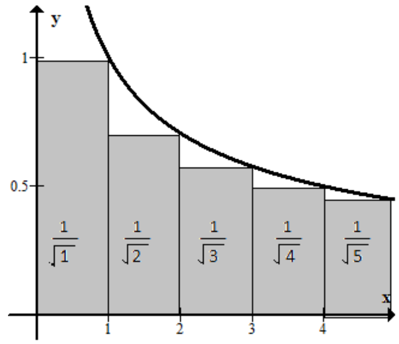
and

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} 2\sqrt{t} - 2\sqrt{1} = \infty$$

So our series must add to more than what the integral sums to and we just showed that it diverges to ∞. Therefore, our series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}} \text{ diverges to } \infty$$

Why did we choose rectangles above the graph of $y = \dfrac{1}{\sqrt{x}}$. Suppose we choose to make our rectangles below $y = \dfrac{1}{\sqrt{x}}$, as shown below.



This shows that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

sums to a number less than

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$$

The problem is $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx = \infty$. So we really can't say anything about the series. Summing to a number less than ∞ doesn't mean anything.

The method we used is called the **Integral Test**. It involves relating the terms of a series to an improper integral.

5.3.1 The Integral Test

Suppose f is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series

$$\sum a_n$$

converges if and only if $\displaystyle\int_1^{\infty} f(x)\, dx$ converges. This means

1. If $\displaystyle\int_1^{\infty} f(x)\, dx$ converges, then $\sum a_n$ converges.

2. If $\int_1^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

Example 5.32. Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2} = \frac{1}{2^2} + \frac{2}{5^2} + \frac{3}{10^2} + \frac{4}{17^2} + \ldots$$

1. Let's first determine if the sequence $a_n = \dfrac{n}{(n^2+1)^2}$ converges to 0.

$$\lim_{n\to\infty} \frac{n}{(n^2+1)^2} \approx \lim_{n\to\infty} \frac{n}{n^4} = 0$$

   Since the sequence $a_n \to 0$, it's **possible** the series converges.

2. Check to make sure it's decreasing. In order to use the **integral test**, the sequence must always be decreasing. We figure this out by looking at the derivative.

   Let $f(x) = \dfrac{x}{(x^2+1)^2}$. Usin the quotient rule to find $f'(x)$, we get

$$f'(x) = \frac{1-3x^2}{(x^2+1)^2}$$

   For sufficiently large x values, $f'(x) < 0$, meaning the original function $f(x)$ is always decreasing.

3. The conditions to use the integral test have been met. Let's use the integral test now.

$$\begin{aligned}
\int_1^\infty \frac{x}{(x^2+1)^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{x}{(x^2+1)^2}\,dx \\
&\quad \text{Let } u = x^2+1, \text{ and } du = 2x\,dx \\
&= \lim_{t\to\infty}\int_2^t \frac{u^{-2}}{2}\,dx \\
&= \lim_{t\to\infty} -\frac{1}{2u}\Big|_2^t \\
&= \lim_{t\to\infty} -\frac{1}{2t}+\frac{1}{4} \\
&= 0+\frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

Since $\displaystyle\int_1^\infty \frac{x}{(x^2+1)^2}\,dx$ converges, the integral test concludes the series

$$\sum_1^\infty \frac{n}{(n^2+1)^2} \text{ converges}$$

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5.3.2 Convergence of p-series

This will be an **extremely** useful series. In later sections, we use this series quite a bit.

Question: What values of p is $\sum \frac{1}{n^p}$ convergent?

It's a pretty simple check. We use the integral test to determine the convergence. First, we need to make sure the conditions have been satisfied to use the integral test.

1. Is $a_n = \frac{1}{n^p} > 0$ for $n \geq 1$? Yes, $a_n > 0$ for all values of p as long as $n \geq 1$.
2. Is a_n decreasing? Let $a_n = f(n) = \frac{1}{n^p}$.

   Since $f'(n) = \frac{-p}{n^{p+1}} < 0$ for $n \geq 1$, we conclude a_n is always decreasing.

3. The conditions are met, so let's use the integral test. For what vaules of p does $\displaystyle\int_1^{\infty} \frac{1}{x^p}\, dx$ converge?

   We showed back in the **Improper Integral** section that

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ 0, & p \leq 1 \end{cases}$$

   Therfore, the p-series, $\displaystyle\sum_1^{\infty} \frac{1}{n^p}$ converges when $p > 1$. Like I said...simple.

Example 5.33. Consider the following p series. Which ones converge?

1. $\displaystyle\sum_1^{\infty} \frac{1}{n^{16}}$. This series converges since $p = 16 > 1$.
2. $\displaystyle\sum_1^{\infty} n^{-1/4}$. First, rewrite it as a p-series.

$$\sum_1^{\infty} \frac{1}{n^{1/4}}$$

   which diverges because $p = \dfrac{1}{4} < 1$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{-3}}$. This series diverges because $p = -3 < 0$.
4. $\displaystyle\sum_{n=1}^{\infty} n^{-1.001} + n^{-3/2}$. First, rewrite it as a p-series.

$$\sum_1^{\infty} \frac{1}{n^{1.001}} + \frac{1}{n^{3/2}}$$

   Since the first p-series has $p = 1.001 > 1$ and the second series has $p = 3/2$, they both converge.

Example 5.34. Determine if the following series converges

$$\sum_{n=1}^{\infty} ne^{-n}$$

1. First, let's determine if ne^{-n} is decreasing and converges to 0.

$$f'(n) = e^{-n} - ne^{-n} = e^{-n}(1-n) < 0 \text{ for all } n > 2$$

So the sequence is decreasing. Now we check to see if it converges to 0.

$$\lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} \overset{LH}{=} \lim_{n=1} \frac{1}{e^n} = 0$$

2. The sequence $a_n = ne^{-n}$ satisfies the requirements to use the integral test. Let's use it now.

$$\int_1^{\infty} xe^{-x}\, dx$$

We use Integration by Parts

$$\text{Let } u = x, \quad dv = e^{-x}\, dx$$

$$du = dx, \quad v = -e^{-x}$$

We'll drop the bounds for now

$$\begin{aligned} \int xe^{-x}\, dx &= -xe^{-x} - \int -e^{-x}\, dx \\ &= -xe^{-x} + \int e^{-x}\, dx \\ &= -xe^{-x} - e^{-x} \end{aligned}$$

Therefore,

$$\begin{aligned}
\int_1^\infty xe^{-x}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x}\,dx \\
&= \lim_{t\to\infty} -xe^{-x} - e^{-x}\Big|_1^t \\
&= \lim_{t\to\infty} -te^{-t} - e^{-t} - \left(-1e^{-1} - e^{-1}\right) \\
&= 0 - 0 - \left(-2e^{-1}\right) \\
&= 2e^{-1}
\end{aligned}$$

You can verify

$$\lim_{t\to\infty} -te^{-t} = 0$$

by using L'Hospitals Rule.

3. Now that we showed the integral converges, the integral test concludes the series

$$\sum_{n=1}^{\infty} ne^{-n} \text{ converges}$$

Example 5.35. Determine if the following series converges

$$\sum_{2}^{\infty} \frac{1}{n\ln n}$$

1. The sequence $a_n = \dfrac{1}{n\ln n}$ decreases and converges to 0. At this point, something like this should be easy to identify.

2. Since the sequence satisfies the requirements for the integral test, let's use it now.

$$\int_2^\infty \frac{1}{x\ln x}\,dx$$

Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$. Using this substitution we have

$$\int_{\ln 2}^\infty \frac{1}{u}\,du = \ln u\big|_{\ln 2}^\infty = \infty - \ln(\ln 2) = \infty$$

3. Since the integral diverges, the Integral Test concludes the following series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges}$$

The last part of this section is to discuss a way to approximate a series. Suppose you find the n-th partial sum S_n. Since this isn't the true sum, there is a remainder amount R_n such that

$$S = \underbrace{a_1 + a_2 + a_3 + a_4 + \ldots + a_n}_{S_n} + \underbrace{a_{n+1} + a_{n+2} + a_{n+3} + \ldots}_{R_n}$$

S_n is probably a decent approximation, but sometimes we want a bit of accuracy. If the sequence a_n satisfies the integral test, then

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \ldots \leq \int_n^{\infty} f(x)\, dx$$

Example 5.36. Find the 5-th partial sum of

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

We add up the first 5 terms of the sequence $a_n = \dfrac{n^2}{e^n}$.

$$S_5 = \frac{1}{e^1} + \frac{4}{e^2} + \frac{9}{e^3} + \frac{16}{e^4} + \frac{25}{e^{25}} = 1.818803087$$

Suppose I want a partial sum that's accurate up to 0.00001. This means

$$R_n \leq 0.00001$$

$$\int_n^{\infty} f(x) = \int_n^{\infty} \frac{x^2}{e^x} = \int_n^{\infty} x^2 e^{-x}\, x \leq 0.00001$$

We need to find $\displaystyle\int_n^{\infty} x^2 e^{-x}\, dx$. Using integration by parts, you can verify that

$$\int_n^{\infty} x^2 e^{-x}\, dx = e^{-n}(n^2 + 2n + 2)$$

Now we just have to solve for n

$$e^n(n^2 + 2n + 2) < 0.00001$$

There really isn't an easy way to solve this. I recommend just try plugging in some values for n.

1. $n = 5$

$$e^5(5^2 + 2(5) + 2) = 0.249304 > 0.00001$$

2. $n = 10$

$$e^{10}(10^2 + 2(10) + 2) = 0.005539 > 0.00001$$

3. $n = 15$

$$e^{15}(15^2 + 2(15) + 2) = 0.000079 > 0.00001$$

4. $n = 16$

$$e^{16}(16^2 + 2(15) + 2) = 0.000033 > 0.00001$$

5. $n = 17$

$$e^{17}(17^2 + 2(17) + 2) = 0.000013 > 0.00001$$

6. $n = 18$

$$e^{18}(18^2 + 2(18) + 2) = 0.000006 < 0.00001$$

Therefore, we need to add up the first 18 terms to be accurate to within 0.00001.

$$\sum_{n=1}^{18} \frac{n^2}{e^n} = \frac{1}{e^1} + \frac{4}{e^2} + \frac{9}{e^3} + ... + \frac{18}{e^{18}} = 1.99229$$

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