5.2 Infinite Series

Since many quantities show up that cannot be computed exactly, we need some way of representing it (or approximating it). One way is to sum an infinite series. Recall that a_n is the sequence $\{a_1, a_2, a_3, a_4, \ldots\}$.

A series is summing up all the terms of an infinite sequence. We use the uppercase sigma to denote the sum.

$$\sum_{n=1}^{n} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$$

We are going to spend a lot of time determining if the series sums to a finite number, or diverges. For example,

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \ldots = \sin(1)$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$

Now it's impossible to add up infinitely many numbers, so we do the following instead.

Definition 5.6 (Partial Sums)**.** A partial sum S_N is the finite sum of the terms up to and including the N-th term. Consider

$$\sum_{n=1}^{N} \frac{1}{2^n}$$

1. $S_1 = \frac{1}{2}$
2. $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
3. $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$
4. $S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$
5. $S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$
6. ...

Do you see how each partial sum is just the sum of the terms up to the N-th term? Do you also see how there's a pattern to the partial sum. The partial sums are increasing but they don't appear to ever get bigger than 1. As it was stated before, the infinite series actually sums to 1.

Most problems you won't be able to find the exact sum. We're mostly concerned with the question, does the sum converge. But in this particular case, we can figure it out.

A list of partial sums is itself a sequence. It's the sequence of partial sums. If we can show the sequence of partial sums converges to a number, then we found its sum. You can actually see a pattern for the partial sums for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$S_n = \frac{2^n - 1}{2^n}$$

which converges nicely.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$$

Definition 5.7 (Series Convergence)**.** If $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \ldots + a_n$ and S_n converges to a finite number S, then $\sum_{1}^{\infty}$ converges and $\sum_{1}^{\infty} = S$

Otherwise, the series is divergent.

5.2.1 Geometric Series

A geometric series has the form $\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^n$. Some examples are

1. $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$, where $r = \frac{1}{2}$.

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$

2. $\sum_{n=0}^{\infty} 3^n$, where $r = 3$

$$\sum_{n=1}^{\infty} 3^n = 1 + 3 + 9 + 27 + 81 + ...$$

3. $\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n = \sum_{n=1}^{\infty} -\frac{2}{3}\left(\frac{-2}{3}\right)^{n-1}$, where $a = -\frac{2}{3}$ and $r = \frac{-2}{3}$

Now (1) and (3) converge but (2) diverges. So the natural question is, "what does r have to be so the series converges?"

We can figure out the formula for the partial sums of a geometric series. Let's go ahead and do that now.

Consider the infinite series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots$.

Let

$$S_N = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots + ar^{N-1}$$

Multiply every term by r

$$rS_N = ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + \ldots + ar^N$$

Now consider

$$S_N - rS_N = a - ar^N$$

Note that all the middle terms will cancel when you subtract the two. Now we just solve for S_N.

$$S_N(1-r) = a - ar^N$$

$$S_N = \frac{a - ar^N}{1-r} = \frac{a(1-r^N)}{1-r}$$

Does this partial sum formula work? Let's go back and consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$. It needs to be written in the correct form. It now looks like

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

So $a = \dfrac{1}{2}$ and $r = \dfrac{1}{2}$. We know $S_5 = \dfrac{31}{32}$ from a while back. Let's see if our formula for partial sums of a geometric series works.

$$S_5 = \frac{\frac{1}{2}(1 - \left(\frac{1}{2}\right)^5)}{1 - \frac{1}{2}} = \frac{31}{32}$$

Warning: Some textbooks define $S_N = \dfrac{a(1-r^{N+1})}{1-r}$. It depends on how they originally defined S_N. In this case, $S_N = a + ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{N-1} + ar^N$. Do you see how they went one more term than we did? They have that ar^N. Since they go one more term than us, they have r^{N+1} in their formula.

Even though a partial sum will exist for every geometric series, the infinite sum may not. It depends on r.

Theorem 5.2 (The Sum of a Geometric Series)**.** *Let $a \neq 0$. If $-1 < r < 1$, then*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ar^4 + \ldots = \frac{a}{1-r}$$

$$\sum_{n=0}^{\infty} ar^{n} = a + ar + ar^2 + ar^3 + ar^4 + \ldots = \frac{a}{1-r}$$

If $|r| > 1$, then the geometric series diverges.

Something to note,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n}$$

Let's do some examples:

Example 5.18. Find

$$\sum_{n=1}^{\infty} \frac{2}{3^{n-1}}$$

Make sure it's in one of the above forms, which this is not.

$$\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = \sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$$

So $a = 2$ and $r = \dfrac{1}{3}$. Therefore,

$$\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1} = \frac{2}{1-\frac{1}{3}} = 3$$

Example 5.19. Find

$$\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^{n}$$

It's not in one of the above forms. It's possible to get it in that form, but there's an easier way. We use a modified version of the geometric series formula. This is what we can use when the series begins at $n = M$. In this case, it begins at $M = 3$.

$$\sum_{n=M}^{\infty} ar^n = \frac{ar^M}{1-r}$$

$$\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n$$

So $a = 7$ and $r = \dfrac{-3}{4}$ and $M = 3$. Therefore,

$$\sum_{n=3}^{\infty} 7\left(-\frac{3}{4}\right)^n = \frac{7\left(\frac{-3}{4}\right)^3}{1-\frac{-3}{4}} = -\frac{27}{16}$$

Truthfully, I've never used this formula until the writing of these notes. So I'm going to show you how I would have done it without the modified geometric series formula.

I try to adjust the series so it starts at $n = 1$ and uses the exponent $n - 1$.

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or make the series begin at $n = 0$, but use the exponent n.

$$\sum_{n=0}^{\infty} ar^n$$

Here's a trick about adjusting where a series begins. Every time you start the series one step earlier, (in this case, $n = 2$), you must raise the exponent in the formula by 1.

$$\sum_{n=3}^{\infty} ar^n = \sum_{n=2}^{\infty} ar^{n+1} = \sum_{n=1}^{\infty} ar^{n+2} = \sum_{n=0}^{\infty} ar^{n+3}$$

So we can rewrite our series as any of the following,

$$\sum_{n=3}^{\infty} 7\left(\frac{-3}{4}\right)^n = \sum_{n=2}^{\infty} 7\left(\frac{-3}{4}\right)^{n+1} = \sum_{n=1}^{\infty} 7\left(\frac{-3}{4}\right)^{n+2} = \sum_{n=0}^{\infty} 7\left(\frac{-3}{4}\right)^{n+3}$$

I'll use

$$\sum_{n=1}^{\infty} 7\left(\frac{-3}{4}\right)^{n+2}$$

So now I just have to adjust the exponent so its $n-1$. Note that

$$\left(\frac{-3}{4}\right)^{n+2} = \left(\frac{-3}{4}\right)^3 \cdot \left(\frac{-3}{4}\right)^{n-1}$$

So my final series is

$$\sum_{n=3}^{\infty} 7\left(\frac{-3}{4}\right)^n = \sum_{n=1}^{\infty} 7\left(\frac{-3}{4}\right)^3 \cdot \left(\frac{-3}{4}\right)^{n-1}$$

Technically, our $a = 7\left(\frac{-3}{4}\right)^3$ and $r = \frac{-3}{4}$. The series sums to

$$\frac{a}{1-r} = \frac{7\left(\frac{-3}{4}\right)^3}{1-\left(\frac{-3}{4}\right)} = -\frac{27}{16}$$

I know that was a lot of work. But it gets easier if you stick with it.

Example 5.20. Find

$$\sum_{n=0}^{\infty} 5^{-n}$$

We need to rewrite the sequence 5^{-n} as $\frac{1}{5^n} = \left(\frac{1}{5}\right)^n$. Now you can see $a = 1$ and $r = \frac{1}{5}$. Therefore,

$$\sum_{n=0}^{\infty} 5^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$$

Example 5.21. Find

$$\sum_{n=0}^{\infty} \frac{2+3^{n+1}}{5^n}$$

The trick here is to split up the sequence $\dfrac{2+3^{n+1}}{5^n}$ as $\dfrac{2}{5^n} + \dfrac{3^{n+1}}{5^n} = 2\left(\dfrac{1}{5}\right)^n + 3\left(\dfrac{3}{5}\right)^n$, and compute them separately.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2+3^{n+1}}{5^n} &= \sum_{n=0}^{\infty} 2\left(\frac{1}{5}\right)^n + \sum_{n=0}^{\infty} 3\left(\frac{3}{5}\right)^n \\ &= \frac{2}{1-\frac{1}{5}} + \frac{3}{1-\frac{3}{5}} \\ &= 10\end{aligned}$$

Example 5.22. Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + ...$$

Since it's a geometric series, we need to find a and r. a is always the first term in the sequence, so $a = 5$. The second term is ar.

$$ar = 5r = -\frac{10}{3}$$

Solving for r, we get $r = -\dfrac{2}{3}$. Therefore, the geometric series sums to

$$\sum_{n=1}^{\infty} 5\left(\frac{-2}{3}\right)^{n-1} = \frac{5}{1-\left(\frac{-2}{3}\right)} = 3$$

Example 5.23. Find

$$\sum_{n=1}^{\infty} \left(\frac{3}{11}\right)^{-n}$$

You should rewrite the series so the exponent is positive.

$$\sum_{n=1}^{\infty}\left(\frac{11}{3}\right)^n$$

This series is geometric with $r = \frac{11}{3}$. Since $r > 1$, the series diverges to ∞.

5.2.2 Telescoping Series

Consider the following series:

$$\sum_{n=1}^{\infty}\frac{2}{n(n+1)}$$

.

Let's check out some partial sums.

1. $S_1 = \frac{2}{2} = 1$
2. $S_2 = \frac{2}{2} + \frac{2}{2\cdot 3} = \frac{4}{3}$
3. $S_3 = \frac{2}{2} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} = \frac{3}{2}$
4. $S_4 = \frac{2}{2} + \frac{2}{2\cdot 3} + \frac{2}{3\cdot 4} + \frac{2}{4\cdot 5} = \frac{8}{5}$
5. Let's skip ahead a bit
6. $S_{10} = \frac{20}{11}$
7. $S_{50} = \frac{100}{51} = 1.96078$
8. $S_{100} = \frac{200}{101} = 1.9802$
9. $S_{1000} = 1.998$.

It appears the sum is approaching 2. But how we can prove that? Recall a while back we learned how to integrate by a method called **Partial Fractions**. The method required us to break up a fraction into smaller fractions. Let's do that here with $\frac{2}{n(n+1)}$.

$$\frac{2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

Multiply everything by $n(n+1)$

$$2 = A(n+1) + B(n)$$

$$2 = (A+B)n + A$$

Just by inspection, we see that $A = 2$, which makes $B = -2$. Let's try the partial sums again.

1. $S_1 = \frac{2}{1} - \frac{2}{2} = 1$

2. $S_2 = \left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) = \frac{2}{1} - \frac{2}{3}$

3. $S_3 = \left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) = \frac{2}{1} - \frac{2}{4}$

4. $S_4 = \left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) = \frac{2}{1} - \frac{2}{5}$

I'm hoping you see a pattern. The n-th partial sum is going to be

$$S_n = \frac{2}{1} - \frac{2}{n+1}$$

since all of the terms but the first and last cancel. And recall that a series sums to a finite number if limit of S_n exists.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{2}{1} - \frac{2}{n+1} = \frac{2}{1} - 0 = 2$$

Example 5.24. Determine if the series

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+...$$

converges.

We need to find the formula a_n for the series $\sum_{n=1}^{\infty} a_n$. This is mainly guess and check. Since n starts at 1, my initial guess is

$$a_n=\frac{1}{n(n+2)}$$

Now that I know I have a telescoping series, I need to use partial fraction decomposition to rewrite a_n.

$$a_n=\frac{1}{2}\left[\frac{1}{n}-\frac{1}{n+2}\right]$$

, which gives us the series

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+2}$$

Let's write out the first few partial sums so we can figure out the partial sum sequence S_n. I pulled the $\frac{1}{2}$ out. We'll deal with it a bit later.

1. $S_1=\frac{1}{1}-\frac{1}{3}$
2. $S_2=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)$
3. $S_3=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}$
4. $S_4=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)=\frac{1}{1}+\frac{1}{2}-\frac{1}{5}-\frac{1}{6}$

5. $S_5 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{1} + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$

It looks like $S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$. The sequences converges to

$$\lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{1} + \frac{1}{2}$$

Therefore,

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2} = \frac{1}{2}\left[1 + \frac{1}{2}\right] = \frac{3}{4}$$

Example 5.25. Find

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right)$$

If you start looking at partial sums right away, you get something like this

1. $S_1 = \ln\left(\frac{2}{3}\right)$
2. $S_2 = \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right)$
3. $S_3 = \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{4}{5}\right)$
4. $S_4 = \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \ln\left(\frac{4}{5}\right) + \ln\left(\frac{5}{6}\right)$

It's hard to see what's happening. Luckily, we use the property of ln() to break up our sequence

$$\ln\left(\frac{n+1}{n+2}\right) = \ln(n+1) - \ln(n+2)$$

Now let's try those partial sums again.

1. $S_1 = \ln(2) - \ln(3)$

2. $S_2 = (\ln(2) - \ln(3)) - (\ln(3) - \ln(4)) = \ln(2) - \ln(4)$

3. $S_3 = (\ln(2) - \ln(3)) - (\ln(3) - \ln(4)) + (\ln(4) - \ln(5)) = \ln(2) - \ln(5)$

4. $S_4 = (\ln(2) - \ln(3)) - (\ln(3) - \ln(4)) + (\ln(4) - \ln(5)) + (\ln(5) - \ln(6)) = \ln(2) - \ln(6)$

If you keep going, we find the partial sums have the form

$$S_n = \ln(2) - \ln(n+2)$$

Finding the limit we see that

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} \ln(2) - \ln(n+2) = \ln(2) - \infty = -\infty$$

The next theorem is extremely useful.

Theorem 5.3. *Consider the series* $\sum a_n$. *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum a_n$ *diverges.*

If $\sum a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

So what does this theorem mean? If you can see the sequence a_n does not converge to 0, the series automatically diverges. If you see that a_n does converge to 0, it just means the series has the potential to converge. At that point, we would need to use some other technique to determine if the series converges.

Example 5.26. Consider the following two series.

1. $\sum \dfrac{2}{n(n+1)}$.

   The sequence $a_n = \dfrac{2}{n(n+1)}$ converges to 0 and we already showed the series converges.

2. $\sum \ln\left(\frac{n+1}{n+2}\right)$.

   The sequence $a_n = \ln\left(\frac{n+1}{n+2}\right)$ converges to $\ln(1) = 0$. So the series has the potential to converge. However, we previously showed it does not converge.

Example 5.27. What values of x will the series converge?

$$\sum \frac{x^n}{2^n}$$

Start off by identifying this series is a geometric series. We can rewrite it as

$$\sum \left(\frac{x}{2}\right)^n$$

We know that a geometric series $\sum ar^n$ will converge if $-1 < r < 1$. For our series, $r = \frac{x}{2}$. Solving

$$-1 < \frac{x}{2} < 1$$

we get

$$-2 < x < 2$$

.

Example 5.28. What values of x will the following series converge?

$$\sum 2^n(x+3)^n$$

Just like the previous problem, this series is a geometric series. We have to rewrite it so it has the correct form ar^n.

$$\begin{aligned}\sum 2^n(x+3)^n &= \sum(2(x+3))^n \\ &= \sum(2x+6)^n\end{aligned}$$

So $r = 2x + 6$. Solving

$$-1 < 2x + 6 < 1$$

we get

$$-7 < 2x < -5$$

$$-\frac{7}{2} < x < -\frac{5}{2}$$

.

Therefore, $\sum 2^n(x+3)^n$ converges for $-\frac{7}{2} < x < -\frac{5}{2}$

Example 5.29. Find

$$\sum_{n=1}^{\infty} \sqrt[n]{5}$$

Let's rewrite the sum as

$$\sum_{n=1}^{\infty} 5^{1/n}$$

This may look like a gemetric series but it isn't. It actually doesn't look like anything we've done yet. So if you're not sure where to start, start by checking if $a_n \to 0$. Because if it doesn't, the series diverges.

$$\lim_{n\to\infty} 5^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\cdot\ln(5)} = e^0 = 1 \neq 0$$

Since $a_n \neq 0$, the series $\sum 5^{1/n}$ diverges.

Another interesting use for a geometric series is it can help us change a repeating decimal into its corresponding fraction.

Example 5.30. Change $0.1\overline{72}$ into a fraction.

First, let's write out the fraction as the following

$$0.1\overline{72} = 0.1 + 0.072 + 0.00072 + 0.0000072 + 0.000000072 + \dots$$

Next, change each of the terms into a fraction.

$$0.1\overline{72} = \frac{1}{10} + \frac{72}{10^3} + \frac{72}{10^5} + \frac{72}{10^7} + \frac{72}{10^9} + \frac{72}{10^{11}} + \dots$$

$$0.1\overline{72} = \frac{1}{10} + \frac{72}{10^3}\left(\frac{1}{10^2}\right)^0 + \frac{72}{10^3}\left(\frac{1}{10^1}\right)^1 + \frac{72}{10^3}\left(\frac{1}{10^2}\right)^2 + \frac{72}{10^3}\left(\frac{1}{10^2}\right)^3 + \dots$$

Except for the first term, the rest of the terms form a gemetric series. We can write our decimal as

$$\frac{1}{10} + \sum_{n=0}^{\infty} \frac{72}{10^3}\left(\frac{1}{10^2}\right)^n$$

The geometric piece sums to $\dfrac{72/10^3}{1 - \dfrac{1}{10^2}} = \dfrac{72/10^3}{99/100} = \dfrac{72 \cdot 100}{99 * 10^3} = \dfrac{7200}{99000} = \dfrac{72}{990}$.

So our fraction representation is

$$0.1\overline{72} = \frac{1}{10} + \frac{72}{990} = \frac{99}{990} + \frac{72}{990} = \frac{171}{990}$$

Before moving on to the next section, let's go through the properties for series. You learned this back in calculus I, but it never hurts to review.

1. $\sum c \cdot a_n = c \cdot \sum a_n$
2. $\sum a_n + b_n = \sum a_n + \sum b_n$
3. $\sum a_n - b_n = \sum a_n - \sum b_n$