5.4 The Comparison Test

Goal: We want to compare a series (that's usually complicated) to a series that is known to be convergent or divergent. Let's take a look a couple of easy examples. We want to determine the convergence of

1. $\sum \frac{1}{n^2+1}$

$$\sum \frac{1}{n^2+1} < \sum \frac{1}{n^2} \text{ which converges (p-series)}$$

2. $\sum \frac{1}{\sqrt{n}\cdot 5^n}$

$$\sum \frac{1}{\sqrt{n}\cdot 5^n} < \sum \frac{1}{5^n} \text{ which converges (geometric series)}$$

3. $\sum_{126}^{\infty} \frac{1}{3n^{1/3}-15}$

$$\sum \frac{1}{3n^{1/3}-15} > \sum \frac{1}{3n^{1/3}} = \frac{1}{3}\sum \frac{1}{n^{1/3}} \text{ which diverges (p-series)}$$

In each of these examples, we compared our unknown series to a known series (geometric, p-serires) which we knew converged or diverged. We call this method the **Comparison Tests**.

Definition 5.8 (The Comparison Test)**.** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms such that $0 \leq a_n \leq b_n$ for $n \geq N$.

1. If $\sum b_n$ converges, then $\sum a_n$ also converges.
2. If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Warning: If $\sum b_n$ diverges, then we know nothing of the smaller series $\sum a_n$. If $\sum a_n$ converges, then we know nothing of the larger series $\sum b_n$. Please be careful when using the Direct Comparison Test. Students violate this quite a bit.

So what makes this section difficult? It's finding the known series that works. For example, what would happen if I compared

$$\sum \frac{1}{n^2-1} \text{ and } \sum \frac{1}{n^2}$$

Nothing! Since the first series $\sum \frac{1}{n^2-1} > \sum \frac{1}{n^2}$, we know nothing of the convergence of $\sum \frac{1}{n^2-1}$. But don't worry. We have another test that will deal with this problem later.

So how can we find a sequence that works? Since the sequences have to go to 0,we focus on the denominators. Remember, making a denominator smaller, makes the whole fraction larger. For example,

Example 5.37. $\sum \frac{5}{3n^4+5n+2}$

First, you need to make a guess on its convergence. My guess is it converges. So the next step is to find an easy convergent series whose sequence is larger than $\frac{5}{3n^4+5n+2}$. To do this, we work with the denominator. To find a bigger sequence, we need to make the denominator smaller.

$$3n^4+5n+2 > 3n^4 > n^4$$

So

$$\frac{5}{3n^4+5n+2} < \frac{5}{n^4}$$

Since $\sum \frac{5}{n^4}$ converges (p-test), by the Direct Comparison Test, so does $\sum \frac{5}{3n^4+5n+2}$. By the way, I did not check the to see if the sequence satisfies the conditions to the Direct

Comparison Test. Again, at this point, some of these should be obvious.

Example 5.38. $\sum \frac{(\ln n)^2}{n}$

This one isn't as obvious. Fist, we need to make sure it converges to 0.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(\ln n)^2}{n} &\overset{LH}{=} \lim_{n\to\infty} \frac{2(\ln n)\cdot\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2\ln n}{n} \\
&\overset{LH}{=} \lim_{n\to\infty} \frac{2\cdot\frac{1}{n}}{1} \\
&= \lim_{n\to\infty} \frac{2}{n} \\
&= 0
\end{aligned}$$

Next, let's just verify that $a_n = \frac{(\ln n)^2}{n}$ is decreasing. To show a function is decreasing, we show its derivative is negative. Let $f(n) = \frac{(\ln n)^2}{n}$

$$f'(n) = \frac{n\cdot 2(\ln n)\cdot\frac{1}{n} - (\ln n)^2\cdot 1}{n^2} = \frac{2\ln n - (\ln n)^2}{n^2} = \frac{\ln n(2-\ln(n))}{n^2} < 0$$

for larger values of n.

Now let's find a sequence that we can compare to $\frac{(\ln n)^2}{n}$.

$$\frac{(\ln n)^2}{n} > \frac{1}{n}$$

Since $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n} < \sum \frac{(\ln n)^2}{n}$, the Direct Comparison Test concludes

$$\sum \frac{(\ln n)^2}{n} \text{ diverges}$$

Even though the Direct Comparison Test is nice, the following test will help when an easy comparison can't be made.

Definition 5.9 (Limit Comparison Test)**.** Suppose a_n and b_n are positive sequences. Assume the following limit exists

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

1. If $L > 0$ and is finite, then either both $\sum a_n$ and b_n converge or they both diverge.
2. If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ converges.
3. If $L = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Example 5.39. The following series could not be shown to converge through an obvious Direct Comparison Test

$$\sum \frac{1}{5^n - 1}$$

though I suppose you can show $\dfrac{1}{5^n - 1} < \dfrac{1}{4^n}$ and since $\sum \dfrac{1}{4^n}$ converges, so does $\sum \dfrac{1}{5^n - 1}$. The problem is you have to prove that comparison and that can be time consuming. Instead let's use the Limit Comparison Test.

Let $a_n = \dfrac{1}{5^n - 1}$ and $b_n = \dfrac{1}{5^n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{5^n-1}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{5^n}{5^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{5^n}} = 1$$

Since the limit is finite and $\sum \frac{1}{5^n}$ converges, the Limit Comparison Test concludes

$$\sum \frac{1}{5^n-1} \text{ converges}$$

Example 5.40. Determine the convergence or divergence of

$$\sum \sin(1/n)$$

Since we see $\frac{1}{n}$, we might as well try comparing $\sin(1/n)$ to $\frac{1}{n}$. Note that $\sum \frac{1}{n}$ diverges.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} \stackrel{LH}{=} \lim_{n\to\infty} \frac{\cos(1/n)\cdot -1/n^2}{-1/n^2} = \lim_{n\to\infty} \cos(1/n) = \cos(0) = 1$$

Since the limit value is finite and greater than 0, the Limit Comparison Test concludes

$$\sum \sin(1/n) \text{ diverges}$$

Example 5.41. If $a_n > 0$ and $\sum a_n$ is convergent, show

$$\sum \ln(1+a_n) \text{ converges}$$

They gave us information about a_n, let's try using the Limit Comparison Test with $\ln(1+a_n)$.

$$\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n}$$

Note that as $n \to \infty$ both sequences converge to 0. So this is a L'Hospital Problem. Instead of using a_n, let $f(n) = a_n$. Also note $\lim\limits_{n\to\infty} f(n) = 0$

$$\lim_{n\to\infty} \frac{\ln(1+f(n))}{f(n)} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{1+f(n)} \cdot f'(n)}{f'(n)} = \lim_{n\to\infty} \frac{1}{1+f(n)} = 1$$

By the Limit Comparison Test

$$\sum \ln(1+a_n) \text{ converges}$$

Example 5.42. Show $\sum \frac{n!}{n^n}$ converges.

The trick here is to find a direct comparison to the sequence $a_n = \frac{n!}{n^n}$.

$$\begin{aligned} \frac{n!}{n^n} &= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot \ldots \cdot (n-2)\cdot(n-1)\cdot n}{n\cdot n\cdot n\cdot n\cdot n\cdot n \ldots \cdot n\cdot n\cdot n} \\ &= \frac{2}{n^2}\left(\underbrace{\frac{3}{n}\cdot\frac{4}{n}\cdot\frac{5}{n}\cdot\ldots\cdot\frac{(n-1)}{n}\cdot\frac{n}{n}}_{\text{multiplies to less than 1}}\right) \\ &\leq \frac{2}{n^2} \end{aligned}$$

We now have a direct comparison to the sequence $a_n = \frac{n!}{n^n}$. Since $\sum \frac{1}{n^2}$ converges, by the Direct Comparison Test

$$\sum \frac{n!}{n^n} \text{ converges}$$

Example 5.43. Determine if the following series converges

$$\sum \frac{3n+5}{n(n-1)(n-2)}$$

Note the degree on top is 1 and the degree on the bottom is 3. The difference is 2. So let's compare our sequence to $b_n = \dfrac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{3n+5}{n(n-1)(n-2)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^3+5n^2}{n(n-1)(n-2)} \approx \lim_{n\to\infty} \frac{3n^3}{n^3} = 3$$

Since $L = 3 > 0$ and $\sum \dfrac{1}{n^2}$, the Limit Comparison Test concludes

$$\sum \frac{3n+5}{n(n-1)(n-2)} \text{ converges}$$

Example 5.44. Determine if the following series converges

$$\sum \frac{n}{\sqrt{n^3-n}}$$

Note that

$$\frac{n}{\sqrt{n^3-n}} \approx \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$$

and we know $\sum \dfrac{1}{n^{1/2}}$ diverges (p-series test). Let's use the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^3-n}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3-n}} \approx \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}} = 1$$

Since $L = 1 > 0$ and $\sum \dfrac{1}{n^{1/2}}$ diverges,

$$\sum \frac{n}{\sqrt{n^3 - n}} \text{ diverges}$$

Example 5.45. Determine if $\displaystyle\sum \frac{3}{n! + 6^n}$ converges.

Let's do a direct comparison.

$$\frac{3}{3! + 6^n} < \frac{3}{6^n}$$

Since $\displaystyle\sum \frac{3}{6^n} = \sum \frac{1}{3}\left(\frac{1}{6}\right)^n$ converges (geometric), the direct comparison test concludes

$$\sum \frac{1}{n! + 6^n} \text{ converges}$$

Example 5.46. Determine if $\displaystyle\sum \frac{1}{\sqrt{n} + \ln n}$ converges.

Let's try a limit comparison with $\dfrac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+\ln n}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n} + \ln n}$$

This is a L'Hospital Problem with the type $\dfrac{\infty}{\infty}$.

$$\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{2\sqrt{n}} + \frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1 + \frac{2\sqrt{n}}{n}} = 1$$

That last step I multiplied top and bottom by $2\sqrt{n}$.

Since the limit exists, is finite, and greater than 0, and $\displaystyle\sum \frac{1}{\sqrt{n}}$ diverges, so does $\displaystyle\sum \frac{1}{\sqrt{n} + \ln n}$