5.5 Alternating Series

Most of the tests we've dealt with used a series with positive terms. Now we'll focus on the type of series where the sequence a_n alternates between positive and negative terms.

Definition 5.10 (Alternating Series)**.** A series whose terms are alternately positive and negative.

For example,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$\sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \ldots$$

Theorem 5.4 (Alternating Series Test)**.** *If the alternating series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \ldots$$

satisfies the following requirements

1. $b_n > 0$

2. $b_{n+1} \leq b_n$ *(terms are decreasing)*

3. $\lim_{n\to\infty} b_n = 0$

then the alternating series is convergent. Additionally, we also have the following conclusion

If $\sum_{n=1} (-1)^{n-1} b_n = S$, then

$$0 < S < a_1 \text{ and } S_{2N} < S < S_{2N+1}$$

This conclusion states that if the altnerating series converges, then its sum will always be between consectuve partial sums. Let's take a look at an example.

Example 5.47. Consider the following alternating series

$$\sum_{n=1} \left(\frac{-2}{3}\right)^{n-1}$$

1. Note, this is a geometric series with $r = \dfrac{-2}{3}$, so we know it will converge. It will converge to

$$\sum_{n=1} \left(\frac{-2}{3}\right)^{n-1} = \frac{1}{1-(\frac{-2}{3})} = \frac{3}{5} = 0.6$$

2. Let's look at it now as an alternating series

$$\sum_{n=1} (-1)^{n-1} \left(\frac{2}{3}\right)^{n-1}$$

   where $b_n = \left(\dfrac{2}{3}\right)^{n-1}$. Note that b_n satisfies the requirements for the alternating series test.

   (a) $b_n > 0$

   (b) b_n is decreasing

   (c) $b_n \to 0$ as $n \to \infty$

   By the alternating series test, $\displaystyle\sum_{n=1} (-1)^{n-1} \left(\frac{2}{3}\right)^{n-1}$ converges. But we already knew that.

3. The last part of the alternating series test states a way to estimate the sum by looking at two consecutive partial sums. It states that the true sum of the series will always be

between two consecutive partial sums. Let's check that. Recall the true sum is $S = 0.6$.

(a) $S_1 = 1$

Note that we satisfy $0 < S < 1$

(b) $S_2 = 1 - \frac{2}{3} = \frac{1}{3}$

and

$$\frac{1}{3} < S < 1$$

(c) $S_3 = 1 - \frac{2}{3} + \frac{4}{9} = \frac{7}{9} \approx 0.77778$

and

$$\frac{1}{3} < S < \frac{7}{9}$$

(d) $S_4 = 1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} = \frac{13}{27} = 0.4815$

and

$$\frac{8}{27} < S < \frac{7}{9}$$

4. If we keep this process up, we will eventually trap the true sum between two very close numbers. For example,

$$S_{10} \approx 0.6069$$

$$S_{11} \approx 0.5953$$

or

$$S_{20} \approx 0.60012$$

$$S_{21} \approx 0.59992$$

Example 5.48. Recall that the harmonic series $\displaystyle\sum_{n=1} \frac{1}{n}$ diverges (p-series test). What about

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

This series is an alternating series. Let $b_n = \dfrac{1}{n}$.

1. $b_n = \dfrac{1}{n} > 0$
2. $b_n = \dfrac{1}{n}$ is decreasing
3. $b_n \to 0$ as $n \to \infty$

Since it satisfies the altnerating series test, we conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ converges}$$

Example 5.49. Determine if $\displaystyle\sum_{n=2} \frac{(-1)^n(3n-1)}{2n+1}$ converges.

It's clear it altnernates because of the $(-1)^n$. So now let's consider $b_n = \dfrac{3n-1}{2n+1}$ and see if it satisfies the rest of the alternating series test.

1. $b_n = \dfrac{3n-1}{2n+1} > 0$

2. If we want to check to see if b_n is decreasing, we need to find the derivative and show it's negative. The problem is the derivative is

$$\frac{5}{(2n+1)^2} > 0 \text{ for all } n$$

So b_n is not decreasing.

3. Also, $b_n \to \frac{3}{2} \neq 0$.

The fact that the series is not decreasing to 0 means this series diverges. Recall from a few sections ago that if the sequence a_n doesn't converge to 0 means the series $\sum a_n$ automatically diverges.

Let's return to estimating an alternating series.

Theorem 5.5. *Let* $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$*, where* $b_n > 0$*, decreases, and* $b_n \to 0$*. Then*

$$|R_n| = |S - S_N| < b_{N+1}$$

This theorem states that you can add up the first N terms. The remainder (the stuff that we haven't added yet) must be less than the next term in the sequence b_{N+1}.

Example: Recall the series $\sum_{n=1} (-1)^{n-1} \left(\frac{2}{3}\right)^{n-1}$, which converges to $S = 0.6$.

$$R_{10} = |S - S_{10}| < b_{11} = \left(\frac{2}{3}\right)^{11} = 0.011561$$

Recall from a couple of examples ago that $S_{10} = 0.6069$. So did our theorem work?

$$R_{10} = |0.6 - 0.6069| = 0.0069$$

so

$$R_{10} = 0.0069 < b_{11} = 0.01156$$

Yes, our theorem worked. Let's check another one.

$$R_{20} = |0.6 - 0.60012| = 0.00012$$

$$b_{21} = 0.0002$$

so

$$R_{20} = 0.00012 < b_2 1 = 0.00020$$

So what can we get from this method? If we want to estimate to within say, 0.0001, we need to find out when $|R_N| < 0.0001$ or instead $b_{N+1} < 0.0001$.

Example: Consider the following series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!}$$

Estimate the series so it has an error less than 0.0000005.

1. In order to use our theorem, it needs to satisfiy the altnerating series test. Let $b_n = \dfrac{1}{10^n n!}$.

(a) $b_n = \dfrac{1}{10^n n!} > 0$

(b) b_n is decreasing. We don't have a derivative to check, but c'mon! I think we can agree this decreases.

(c) $b_n \to 0$ as $n \to \infty$

2. Now that we know it converges by the Alternating Series Test, let's work on estimating. Initially we want to find N such that

$$|R_N| < 0.0000005$$

But since $|R_N| < b_{N+1}$, it's much easier to find N where

$$b_{N+1} < 0.0000005$$

So let's get started.

$$\frac{1}{10^{N+1}(N+1)!} < 0.0000005$$

$$2000000 < 10^{N+1}(N+1)!$$

3. We just need to check a few values for N

   (a) If $N = 3$,

$$10^4 4! = 240000 < 2000000$$

   (b) If $N = 4$,

$$10^5 5! = 12000000 > 2000000$$

4. We conclude we need $N \geq 4$ to get our series within 0.0000005. Since n starts at $n = 1$, we need the first $N = 4$ terms to get our required accuracy. Let's verify that we did in fact get our required accuracy. Assume the true sum is $S = 0.095162582$.

   (a) $S_2 = 0.095$

$$|S_2 - S| = 0.0001625 > 0.0000005$$

   (b) $S_3 = 0.09516667$

$$|S_3 - S| = 0.000004078 > 0.0000005$$

(c) $S_4 = 0.09516258$

$$|S_4 - S| = 0.000000002 < 0.0000005$$

5. So it's confirmed. We reach our required accuracy at $N = 4$.