5.6 Absolute Convergence and The Ratio and Root Tests

Recall from our previous section that $\sum \frac{1}{n}$ diverged but $\sum \frac{(-1)^{n-1}}{n}$ converged. Both of these sequences have $b_n = \frac{1}{n}$. Again, one converges and the other does not.

We introduce two terms to distinguish between these two cases. Let

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \dots$$

Definition 5.11 (Absolute Convergence)**.** A series $\sum a_n$ is **Absolutely Convergent** if the series $\sum |a_n|$ is convergent.

A quick note: All convergent series with positive terms are automatically absolutely convergent since

$$\sum |a_n| = \sum a_n$$

Consider the following two examples:

Example 5.50. $\sum \frac{(-1)^{n-1}}{n^2}$

$$\sum \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum \frac{1}{n^2} \text{ which converges}$$

So the series $\sum \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

Example 5.51. $\sum \frac{(-1)^{n-1}}{n}$

Even though $\sum \frac{(-1)^{n-1}}{n}$ converges,

$$\sum \left| \frac{(-1)^{n-1}}{n} \right| = \sum \frac{1}{n} \text{ diverges}$$

So we cannot say it's absolutely convergent. But the original series did converge. We now introduce our second term for convergence.

Definition 5.12 (Contitional Convergence)**.** If $\sum a_n$ converges but $\sum |a_n|$ diverges, then the series $\sum a_n$ is said to be **Conditionally Convergent**

Theorem 5.6. *If a series $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.*

Example 5.52. Determine if $\sum \frac{\cos(n)}{n^2}$ converges or diverges.

The previous theorem states that if it's absolutely convergent, then it converges. Let's check for absolute convergence.

$$\sum \left|\frac{\cos(n)}{n^2}\right| = \sum \frac{|\cos(n)|}{n^2} \leq \sum \frac{1}{n^2}$$

By the Direct Comparison Test, since $\sum \frac{1}{n^2}$ converges, then

$$\sum \left|\frac{\cos(n)}{n^2}\right| \text{ converges}$$

Therefore,

$$\sum \frac{\cos(n)}{n^2} \text{ converges absolutely}$$

We now go over our last two tests to determine convergence or divergence.

5.6.1 Ratio Test

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.

2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum a_n$ is divergent.

3. If $\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive. You need to use a different test.

Example 5.53.

1. Determine if $\sum \dfrac{(-1)^n n^4}{7^n}$ converges.

$$a_{n+1} = \frac{(-1)^{n+1}(n+1)^4}{7^{n+1}}, \text{ and } a_n = \frac{(-1)^n n^4}{7^n}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^4}{7^{n+1}} \cdot \frac{7^n}{(-1)^n n^4}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^4}{n^4} \cdot \frac{7^n}{7^n \cdot 7}\right|$$

$$= \frac{1}{7}\lim_{n\to\infty}\frac{(n+1)^4}{n^4} = \frac{1}{7} < 1$$

Since $L = \dfrac{1}{7} < 1$, the Ratio Test concludes $\sum \dfrac{(-1)^n n^4}{7^n}$ is absolutely convergent.

2. Determine if $\sum \dfrac{n^n}{n!}$ converges.

Since $\lim\limits_{n\to\infty}\dfrac{n^n}{n!} = \infty \neq 0$, we know it diverges. But let's go ahead and show it with the Ratio Test.

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}, \text{ and } a_n = \frac{n^n}{n!}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^n \cdot (n+1)}{n^n} \cdot \frac{n!}{(n+1)\cdot n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

This is a L'Hospital Problem, which I'll let you work on. If you do it correctly, you should get

$$L = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$$

Since $L = e > 1$, the Ratio Test concludes $\sum \frac{n^n}{n!}$ diverges.

3. Use the Ratio Test on $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

   We know $\sum \frac{1}{n}$ diverges and $\sum \frac{1}{n^2}$ converges. Let's see what the Ratio Test tells us.

   (a) $\sum \frac{1}{n}$

   $$a_{n+1} = \frac{1}{n+1}, \text{ and } a_n = \frac{1}{n}$$

   $$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n+1}{n}\right| = 1$$

   The Ratio Test states that if $L = 1$, the test is inconclusive. We would need to use another test to determine its convergence. Good thing we have the p-series test, right?

   (b) $\sum \frac{1}{n^2}$

   $$a_{n+1} = \frac{1}{(n+1)^2}, \text{ and } a_n = \frac{1}{n^2}$$

   $$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| = 1$$

The Ratio Test states that if $L = 1$, the test is inconclusive. We would need to use another test to determine its convergence.

4. Determine if $\sum \frac{1}{(2n)!}$ converges.

$$a_{n+1} = \frac{1}{(2(n+1))!}, \text{ and } a_n = \frac{1}{(2n)!}$$

$$L = \lim_{n\to\infty} \left| \frac{1}{(2(n+1))!} \cdot \frac{(2n)!}{1} \right| = \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)!} \right|$$

The trick here is to write out a few terms of the factorial until it matches up with another factorial.

$$(2n+1)! = (2n+2)(2n+1)! = (2n+2)(2n+1)(2n)!$$

$$L = \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)!} \right| = \lim_{n\to\infty} \left| \frac{(2n)!}{(2n+2)(2n+1)(2n)!} \right| = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0$$

Since $L = 0 < 1$, the Ratio Test concludes $\sum \frac{1}{(2n)!}$ is absolutely convergent.

5.6.2 Root Test

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent.

2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then $\sum a_n$ is divergent.

3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the test is inconclusive. You must use a different test to determine convergence.

The root test is useful when you have a sequence raised to the n-th power in some way,

$$a_n = (b_n)^n$$

Example 5.54.

1. Determine if $\sum \left(\frac{6n-3}{11n+4}\right)^n$ converges.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{6n-3}{11n+4}\right)^n\right|} = \lim_{n\to\infty} \frac{6n-3}{11n+4} = \frac{6}{11} < 1$$

Since $L = \frac{6}{11} < 1$, the Root Test concludes $\sum \left(\frac{6n-3}{11n+4}\right)^n$ is absolutely convergent.

2. Determine if $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ converges.

$$L = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}} = \lim_{n\to\infty} \sqrt[n]{\left(\left(1+\frac{1}{n}\right)^{-n}\right)^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

You evaluate this limit using L'Hospitals Rule. If you do it correctly, you get

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = e^{-1}$$

Since $L = e^{-1} < 1$, the Root Test concludes $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ converges.

3. Is there any value of k that makes $\sum \frac{2^n}{n^k}$ converge? For example, does something like

$$\sum \frac{2^n}{n^{100,000,000,000}} \text{ converge?}$$

Let's go ahead and use the Ratio Test.

$$a_{n+1} = \frac{2^{n+1}}{(n+1)^k}, \text{ and } a_n = \frac{2^n}{n^k}$$

$$L = \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^k} \cdot \frac{n^k}{2^n} \right| = \lim_{n\to\infty} \left| \frac{2n^k}{(n+1)^k} \right| = 2$$

Since $L = 2 > 1$, the Ratio Test concludes $\sum \frac{2^n}{n^k}$ diverges for all values of k.

4. Does $\sum \frac{1}{\ln n}$ converge?

Since we're still in the Ratio and Root Test section, we might as well use it. Since it doesn't have a power of n, we'll use the Ratio Test.

$$a_{n+1} = \frac{1}{\ln(n+1)}, \text{ and } a_n = \frac{1}{\ln n}$$

$$L = \lim_{n\to\infty} \left| \frac{1}{\ln(n+1)} \cdot \frac{\ln(n)}{1} \right| = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} \overset{LH}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Since $L = 1$, the Ratio Test is inconclusive. We need another test. We can do a Direct Comparison with $\sum \frac{1}{n}$. Since $\ln n < n$ for all $n \geq 1$, then $\frac{1}{\ln n} > \frac{1}{n}$ for all $n \geq 1$.

Since $\sum \frac{1}{n}$ diverges, by the Direct Comparison Test, so does $\sum \frac{1}{\ln n}$.