Math 230

Calculus II

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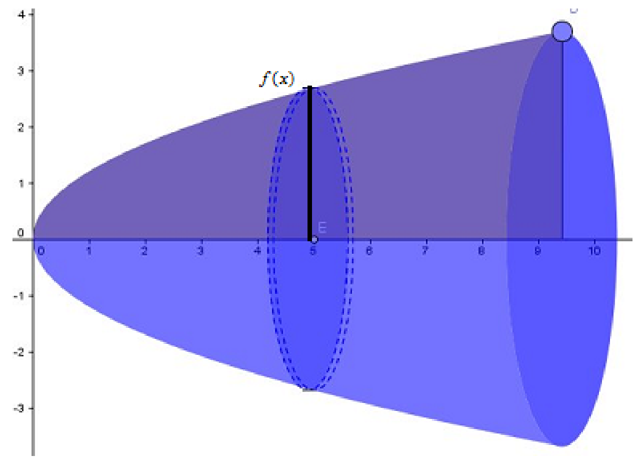
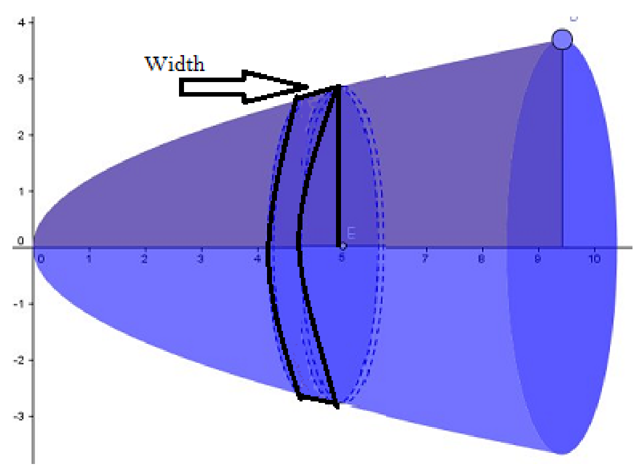
Surface Area

Suppose you want to find the surface area of the following volume.



To find the surface area, we find the circumference of the disk, which is $2\pi r = 2\pi f(x)$. But the width of the disk is not just dx.



The width is actually the length of the line segment. Since we just covered arc length we know how to find the length of a line segment.

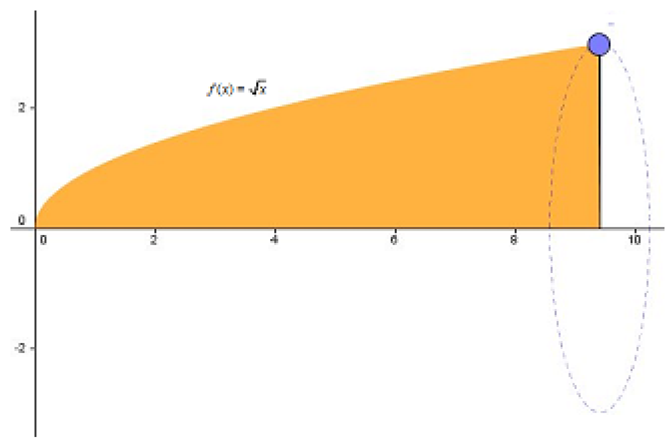
Formula 1: Formula for Surface Area of a Solid of Revolution

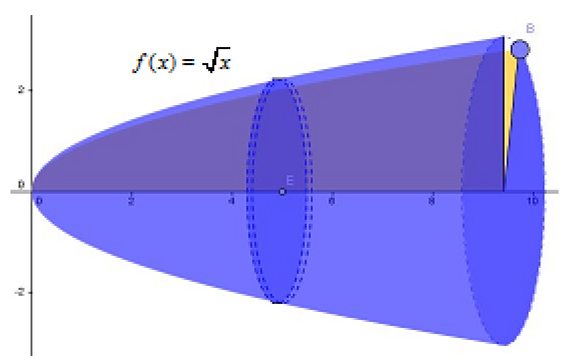
$$S = \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 1

Find the area of the surface genearted by rotating the curve $f(x) = \sqrt{x}$ over the interval [0,9] about the x-axis.

1. Let's go ahead and draw a picture.





2. We need $f(x)$ and $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

   (a) $f(x) = \sqrt{x}$

(b) $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

(c) $\left(\dfrac{dy}{dx}\right)^2 = \dfrac{1}{4x}$

3. Write the formula for the surface area

$$\begin{aligned} S &= \int_0^9 2\pi f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_0^9 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx \\ &= \pi\int_0^4 \sqrt{4x+1}\,dx \end{aligned}$$

The last step was obtained by multiplying $2\sqrt{x}$ and $\sqrt{1+\dfrac{1}{4x}}$ together.

4. Use substitution

(a) Let $u = 4x + 1$

(b) $du = 4\,dx \to \dfrac{du}{4} = dx$

(c) Find new limits of integration

$$\text{If } x = 0, u = 1$$

$$\text{If } x = 9, u = 37$$

(d) Make substitution

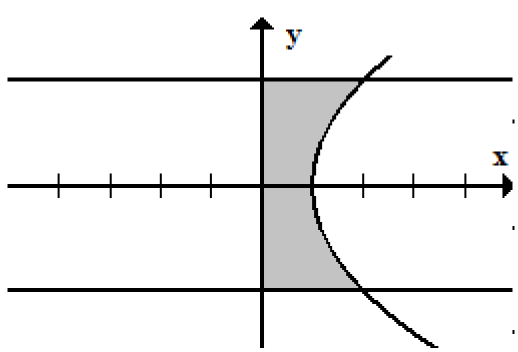
$$\begin{aligned} S &= \frac{\pi}{4}\int_1^{37} u^{1/2}\,du \\ &= \frac{\pi}{6}\,u^{3/2}\Big|_1^{37} \\ &= 37.3437\pi \end{aligned}$$

What about if we revolve around the y-axis? Is anything different? Not really. It just means we're slicing alone the y axis and so we integrate with respect to y.

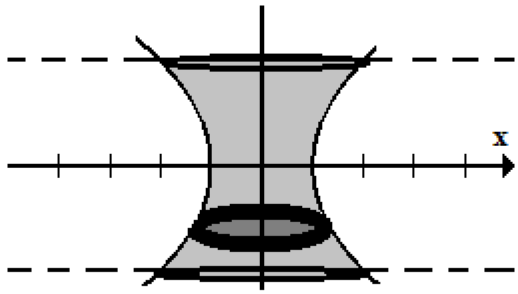
Example 2

Rotate the region bounded by $x = 1+2y^2$, $x = 0$, $y = -1$ and $y = 1$ around the y-axis and setup the integral for the surface area.

1. Let's draw a picture



This is what it looks like after rotating about the y-axis.



2. The radius is $f(y) = 1 + 2y^2$

3. The formula for the surface area is

$$\begin{aligned} S &= \int_{-1}^{1} 2\pi f(y)\sqrt{1+f'(y)^2}\,dy \\ &= \int_{-1}^{1} 2\pi(1+2y^2)\sqrt{1+(4y)^2}\,dy \end{aligned}$$

Example 3

Let $y = x^{-2}$, $1 \le x \le 2$. Setup the integral (in terms of x and y) for the surface after rotating it around the x-axis.

1. The general formula is

$$\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

where in the first formula we must write y in terms of x.

2. Integral in terms of x

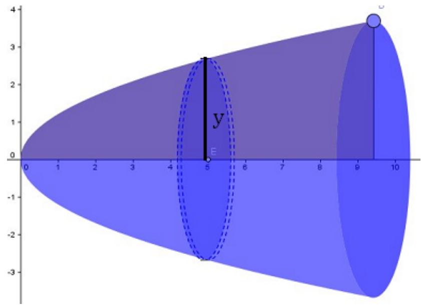
$$\int_1^2 2\pi x^{-2}\sqrt{1+(-2x^{-3})^2}\,dx$$

3. Integral in terms of y

$$\int_{1/4}^{1} 2\pi y\sqrt{1+\left(-\frac{1}{2}y^{-3/2}\right)^2}\,dy$$

Formula 2: Surface Area Formulas

1. **Rotated around the x-axis**



$$SA = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad\qquad SA = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The first integral is in terms of x. That means the y in front of the square root should be in terms of x. For example, $y = x^2 + 2$. You use $x^2 + 2$ instead of y.

2. **Rotated around the y-axis.**

The formula doesn't change much.

$$SA = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad\qquad SA = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The second integral is in terms of y. That means the x in front of the square root should be in terms of y. For example, if $y = x^2$, solve for x ($x = \sqrt{y}$) and use $\sqrt{y}$ instead of x.