Math 230

Calculus II

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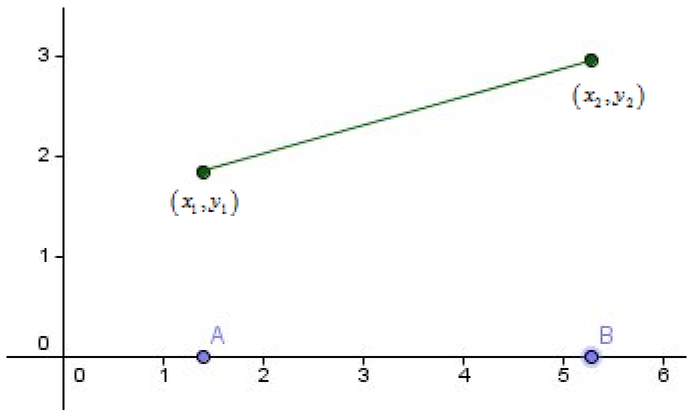
Arc Length

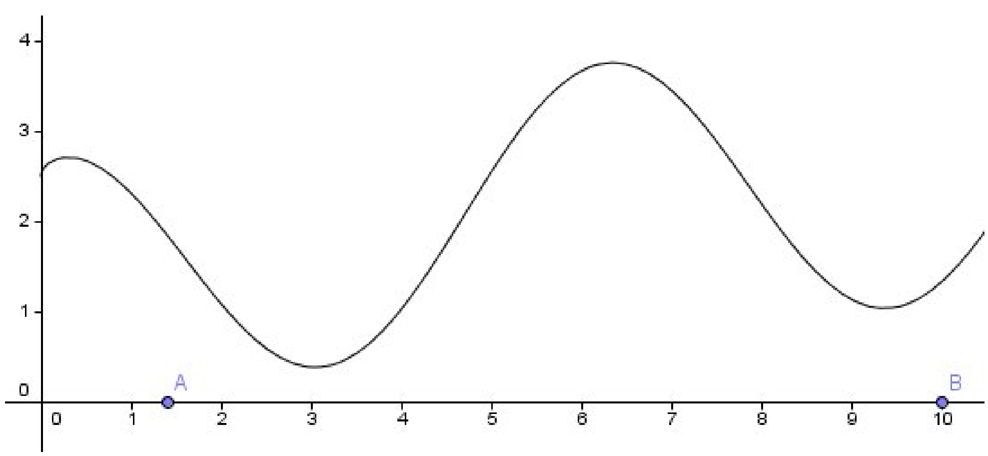
Recall the length of a line segment:



The distance of this line segment is the distance between (x_1, y_2) and (x_2, y_2). We use the distance formula,

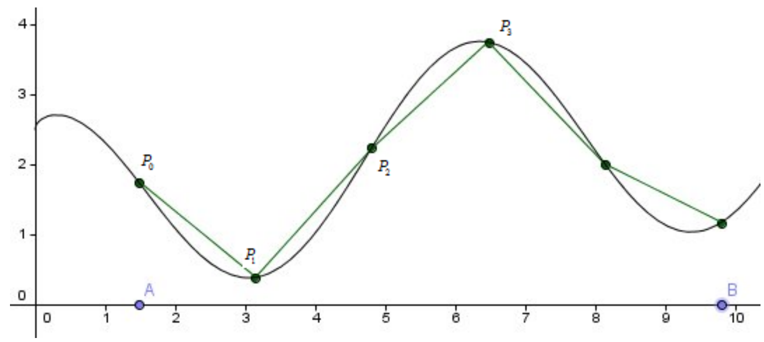
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

But what about something like this? How can we find how long this curve is?

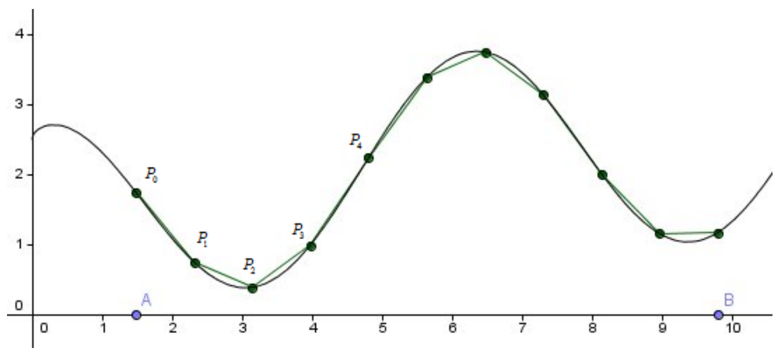


1. The idea is to divide the interval into n equal subintervals each with width Δx.
2. Find the length between the lines connecting the y-values (P_1P_2, P_2P_3, etc.).
3. The length of the curve in each subinterval is approximately the length of each line segment.
4. Add all those lengths together to get an approximate arc length.

$$L \approx \sum_{i=1}^{n} P_{i-1}P_i$$



5. The larger n gets, the better the approximation.

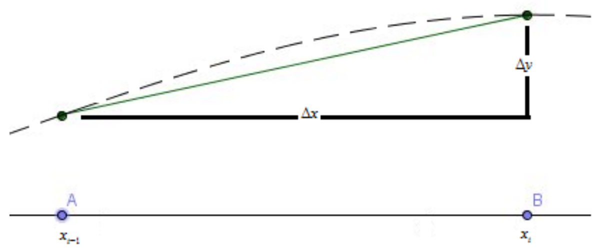


6. Therefore, the exact length is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} P_{i-1}P_i$$

So how do we go from a limit to using an integral to calculate arc length.

Consider the following picture that contains the length of one segment.



1. Find the length again

$$d = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

2. By the Mean Value Theorem, we know the interval $[x_{i-1}, x_i]$, there is a point x_i^* such that

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\Delta x_i$$

We can now write the length of the line segment as

$$\begin{aligned} P_{i-1}P_i &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x_i^2 + \Delta y_i^2} \\ &= \sqrt{\Delta x_i^2 + \left[f'(x_i^*)\right]^2 \Delta x_i^2} \\ &= \sqrt{\Delta x_i^2 \left(1 + \left[f'(x_i^*)\right]^2\right)} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2} \cdot \Delta x_i \end{aligned}$$

3. The total length,

$$L = \lim_{n\to\infty} \sum_{i=0}^{n} \sqrt{1 + f'(x_i^*)^2} \cdot \Delta x$$

which we rewrite as

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

or

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If you're given a function in terms of y, $x = h(y)$, the formula would be

Formula 1: Arc Length Formula

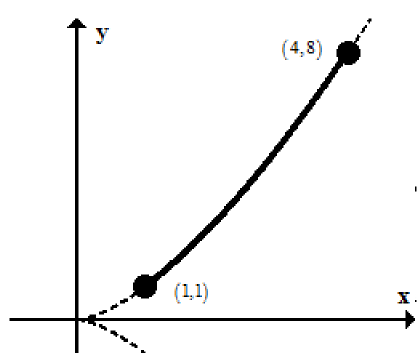
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \text{ if in terms of } x$$

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \text{ if in terms of } y$$

Example 1

Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points (1,1) and (4,8).

1. To graph it, rewrite it as $y = x^{3/2}$.



2. Find $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2}{3}x^{1/2}$$

3. Use the Arc Length Formula

$$\begin{aligned} L &= \int_1^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx \end{aligned}$$

We need to use substitution to finish integrating.

(a) Let $u = 1 + \dfrac{9}{4}x$

(b) $du = \dfrac{9}{4}x \rightarrow \dfrac{4}{9}du = dx$

(c) Change the limits of integration

$$\text{If } x = 1, u = 13/4$$

$$\text{If } x = 4, u = 10$$

(d) Integrate

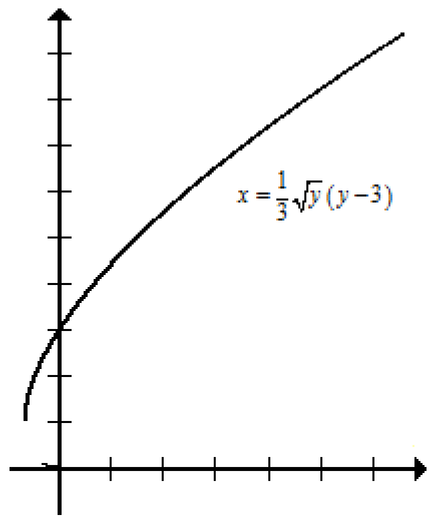
$$\begin{aligned} \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx &= \frac{4}{9}\int_{13/4}^{10} u^{1/2}\, du \\ &= \frac{4}{9}\cdot\frac{2}{3}\, u^{3/2}\Big|_{13/4}^{10} \\ &= 7.63371 \end{aligned}$$

Example 2

Find the exact length of the curve $x = \dfrac{1}{3}\sqrt{y}(y-3)$ over the interval [1,9].

1. This is a function in terms of y. In order to graph this, you'll have to use the techniques from Chapter 1.

2. The graph



3. Find $\dfrac{dx}{dy}$

$$\begin{aligned}
\frac{dx}{dy} &= \frac{1}{3}y^{1/2}\cdot(1) + (y-3)\cdot\frac{1}{6}y^{-1/2} \\
&= \frac{1}{3}y^{1/2} + \frac{1}{6\sqrt{y}}(y-3) \\
&= \frac{1}{6\sqrt{y}}\left(2y + (y-3)\right) \\
&= \frac{1}{6\sqrt{y}}(3y-3) \\
&= \frac{(y-1)}{2\sqrt{y}}
\end{aligned}$$

4. Find $1+\left(\frac{dx}{dy}\right)^2$

$$
\begin{aligned}
1+\left(\frac{y-1}{2\sqrt{y}}\right)^2 &= 1+\frac{(y-1)^2}{4y} \\
&= \frac{4y}{4y}+\frac{y^2-2y+1}{4y} \\
&= \frac{y^2+2y+1}{4y} \\
&= \frac{(y+1)^2}{4y}
\end{aligned}
$$

5. Set up the integral

$$
\begin{aligned}
L &= \int_1^9 \sqrt{1+\left(\frac{y-3}{2\sqrt{y}}\right)^2}\,dy \\
&= \int_1^9 \sqrt{\frac{(y+1)^2}{4y}}\,dy \\
&= \int_1^9 \frac{y+1}{2y^{1/2}}\,dy \\
&= \int_1^9 \frac{1}{2}y^{1/2}+\frac{1}{2}y^{-1/2}\,dy \\
&= \left.\frac{1}{3}y^{3/2}+y^{1/2}\right|_1^9 \\
&= (9+3)-(\frac{1}{3}+1) \\
&= 10.6667
\end{aligned}
$$