Math 230

Calculus II

Brian Veitch • Fall 2015 • Northern Illinois University

Improper Integrals

Well you've made it through all the integration techniques. Congrats! Unfortunately for us, we still need to cover one more integral. They are called **Improper Integrals**.

At this point, we've only dealt with integrals of the form

$$\int_a^b f(x)\, dx$$

Before we talk about the improper type, let's try to build up to it.

Consider the integral

$$\int_1^t \frac{1}{x^2}\, dx$$

Evaluating the integral, we get

$$\int_1^t \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_1^t = -\frac{1}{t} + 1 = 1 - \frac{1}{t}$$

Does this work for any $t > 1$?

1. $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$

2. $\displaystyle\int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{10} = -\frac{1}{10}+1 = \frac{9}{10}$

3. $\displaystyle\int_1^{100} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{100} = -\frac{1}{100}+1 = \frac{99}{100}$

4. $\displaystyle\int_1^{1000} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{1000} = -\frac{1}{1000}+1 = \frac{999}{1000}$

5. $\displaystyle\int_1^{50000} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{50000} = -\frac{1}{50000}+1 = \frac{49999}{50000}$

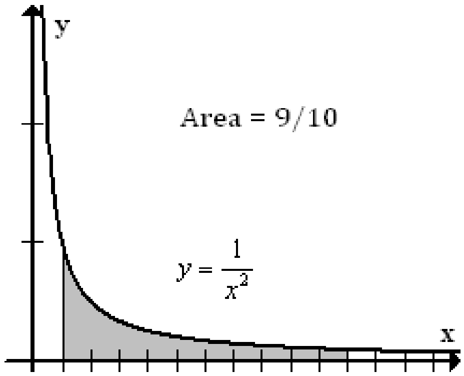
It appears when b gets very large, the value of the integral approaches 1. Notice that

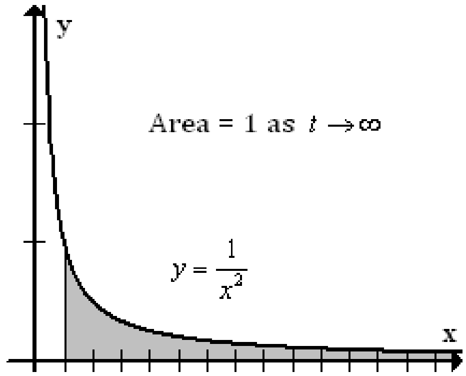
$$\lim_{t\to\infty} -\frac{1}{t}+1 = 1$$

What this says is as t gets very large, say

$$\int_1^{100000000} \frac{1}{x^2}\,dx$$

then the value of that integral is extremely close to 1. And as $t \to \infty$, the value of the integral **converges** to 1.





Definition 1: Definition of an Improper Integral, Type I

1. If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

2. If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

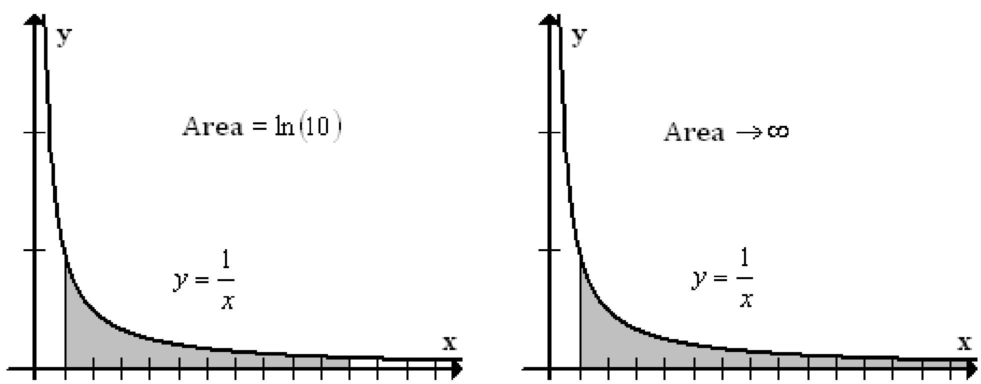
3. If $\int_a^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ exist, they are called **convergent**. If they do not exist, we call them **divergent**.

Example 1

Find $\int_1^\infty \frac{1}{x}\,dx$

Recall, we saw $\int_1^\infty \frac{1}{x^2}\,dx = 1$. Let's see about $\frac{1}{x}$.

$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx\\ &= \lim_{t\to\infty} \ln(x)\big|_1^t\\ &= \lim_{t\to\infty} \ln(t)-\ln(1)\\ &= \lim_{t\to\infty} \ln(t)\\ &= \infty\end{aligned}$$



This is interesting. We changed the degree of the denominator slightly, and now the integral **diverges**. The natural question at this point would be, what does p have to be so

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ exists?}$$

I'm warning you ahead of time that we do a bit of algebra hocus pocus here. Nothing too bad though.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx\\ &= \lim_{t\to\infty}\int_1^t x^{-p}\,dx\\ &= \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right|_1^t\\ &= \lim_{t\to\infty}\frac{1}{1-p}\cdot\frac{1}{x^{p-1}}\Bigg|_1^t\\ &= \lim_{t\to\infty}\frac{1}{1-p}\left[\frac{1}{t^{p-1}}-1\right]\end{aligned}$$

If $p > 1$, then $p - 1 > 0$, so $t^{p-1} \to \infty$. Therefore, we have $\frac{1}{t^{p-1}} \to 0$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$$

What happens when $p < 1$? We already know what happens when $p = 1$.

If $p < 1$, then $p - 1 < 0$. This makes $t^{p-1} \to 0$ as $t \to \infty$. Therefore, $\frac{1}{t^{p-1}} \to \infty$ which makes

$$\int_1^\infty \frac{1}{x^p}\,dx = \infty \text{ (divergent)}$$

If you don't see why $t^{p-1} \to \infty$ when $t \to \infty$, give yourself an example. Let $p = 0.5$.

$$\lim_{t\to\infty} t^{0.5-1} = \lim_{t\to\infty} t^{-1/2} = \lim_{t\to\infty}\frac{1}{t^{1/2}} = 0$$

Which means,

$$\lim_{t\to\infty}\frac{1}{t^{-.5}} = \lim_{t\to\infty} t^{0.5} = \infty$$

This particular integral will be extremely important for the rest of the semester. So **memorize it**!

Example 2

$$\int_{-\infty}^{0} xe^x \, dx$$

Note, all of these integrals will have limits. Many of them will require L'Hospitals Rule.

$$\int_{-\infty}^{0} xe^x \, dx \quad = \quad \lim_{t \to -\infty} \int_{t}^{0} xe^x \, dx$$

And...this one requires Integration by Parts.

1. Let $u = x$, $dv = e^x \, dx$
2. So $du = dx$, $v = e^x$

$$\begin{aligned} \lim_{t \to -\infty} \int_{t}^{0} xe^x \, dx &= \lim_{t \to -\infty} xe^x - \int_{t}^{0} e^x \, dx \\ &= \lim_{t \to -\infty} xe^x - e^x|_t^0 \\ &= \lim_{t \to -\infty} (-0e^0 - e^0) - (te^t - e^t) \\ &= \lim_{t \to -\infty} e^t - te^t - 1 \end{aligned}$$

3. We know $\lim_{t \to -\infty} e^t = 0$
4. We need to find $\lim_{t \to -\infty} -te^t$

$$\begin{aligned}\lim_{t\to-\infty} -te^t &= \lim_{t\to-\infty} \frac{-t}{e^{-t}} \\ &\overset{LH}{\to} \lim_{t\to-\infty} \frac{-1}{-e^{-t}} \\ &\to \frac{-1}{-\infty} \\ &= 0\end{aligned}$$

5. So our final answer is

$$\int_{-\infty}^{0} xe^x \, dx = \lim_{t\to-\infty} e^t - te^t - 1 = -1$$

Seems like we're going to have fun in this section!

Example 3

$\displaystyle\int_1^\infty \frac{x+1}{x^2+2x} \, dx.$

Let's set it up with the limit notation.

$$\int_1^\infty \frac{x+1}{x^2+2x} \, dx = \lim_{t\to\infty} \int_1^t \frac{x+1}{x^2+2x} \, dx$$

For now, we'll drop the $\lim\limits_{t\to\infty}$ until after we anti-differentiate. Also, the integral requires a basic u-substitution.

1. Let $u = x^2 + 2x$
2. $du = 2x + 2 \, dx \to \frac{1}{2} \, du = x + 1 \, dx$
3. Change the bounds

$$\text{If } x = t, \text{ then } u = t^2 + 2t$$

$$\text{If } x = 1, \text{ then } u = 3$$

4. Substitute!

$$\begin{aligned} \int_1^t \frac{x+1}{x^2+2x}\,dx &= \frac{1}{2}\int_3^{t^2+2t} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|u|\Big|_3^{t^2+2t} \end{aligned}$$

Let's get that limit back in.

$$\begin{aligned} \lim_{t\to\infty}\int_3^{t^2+2t}\frac{1}{u}\,du &= \lim_{t\to\infty}\ln|u|\Big|_3^{t^2+2t} \\ &= \lim_{t\to\infty}\ln|t^2+2t| - \ln|3| \\ &= \infty - \ln|3| \\ &= \infty \end{aligned}$$

Example 4

$$\int_0^\infty \frac{dx}{(x+3)(x+4)}$$

We need to use partial fractions.

$$\frac{1}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}$$

1. Start by multiply both sides by $(x+3)(x+4)$

$$1 = A(x+4) + B(x+3)$$

2. Distribute and collect like terms

$$1 = (A+B)x + (4A+3B)$$

3. Match the coefficients

$$\begin{aligned} A+B &= 0 \\ 4A+3B &= 1 \end{aligned}$$

4. Solving this sytem, we get $A = 1$ and $B = -1$

5. Rewrite the integral

$$\begin{aligned} \int_0^\infty \frac{1}{x+3} - \frac{1}{x+4}\,dx &= \lim_{t\to\infty} \int_0^t \frac{1}{x+3} - \frac{1}{x+4}\,dx \\ &= \lim_{t\to\infty} \left[\ln|x+3| - \ln|x+4|\right]_0^t \\ &= \lim_{t\to\infty} \ln\left|\frac{x+3}{x+4}\right|_0^t \\ &= \lim_{t\to\infty} \ln\left|\frac{t+3}{t+4}\right| - \ln\left|\frac{3}{4}\right| \\ &= 0 - \ln\left|\frac{3}{4}\right| \\ &= \ln\left|\frac{4}{3}\right| \end{aligned}$$

Example 5

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$$

We don't know how to do this when both bounds are $\pm\infty$. Recall the following property of integrals

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where $a < c < b$

We use this property now. Choose a number (a nice one) between $(-\infty, \infty)$. How about 0? Rewrite the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

Let's do each integral separately.

1. $\displaystyle\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to-\infty} \tan^{-1}(x)\Big|_t^0 \\
&= \lim_{t\to-\infty} \tan^{-1}(0) - \tan^{-1}(t) \\
&= 0 - (-\pi/2) \\
&= \pi/2
\end{aligned}$$

2. $\displaystyle\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2}\,dx &= \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to\infty} \tan^{-1}(x)\Big|_0^t \\
&= \lim_{t\to\infty} \tan^{-1}(t) - \tan^{-1}(0) \\
&= \pi/2 - 0 \\
&= \pi/2
\end{aligned}$$

3. Final Answer

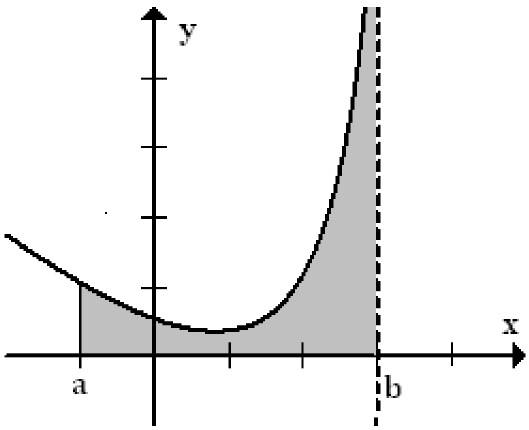
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi/2 + \pi/2 = \pi$$

We conclude the type of integral where ∞ is a bound. Now we move on to the second type of improper integrals.

Definition 2: Type 2 Improper Integrals

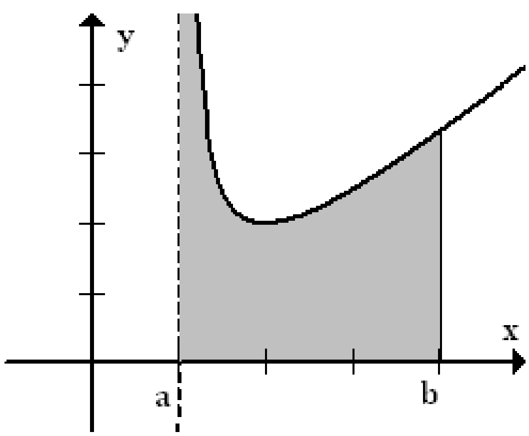
This type of improper integral involves integrals where a bound is where a vertical asymptote occurs, or when one exists in the interval.

1. If f is continuous at $[a, b)$ but discontinuous at b, then



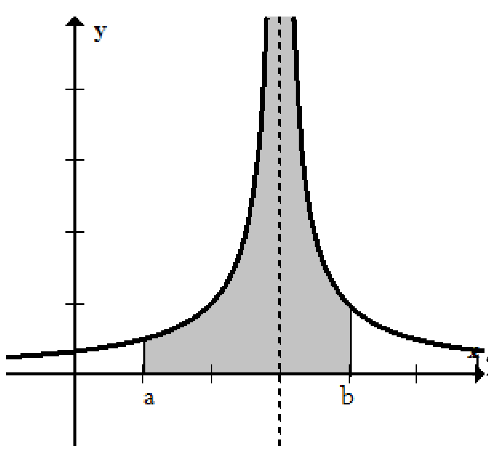
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$
,

2. If f is continuous at $(a, b]$ but discontinuous at a, then



$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$
,

3. If f has a discontinuity at $x = c$, where $a < c < b$ then



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
,

Example 6

Find $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,dx$

This integral is improper because we have an infinite discontinuity (asymptote) at $x = 1$. So we use (1) from above.

$$\begin{aligned}
\int_0^1 \frac{dx}{\sqrt{1-x^2}}\,dx &= \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}}\,dx \\
&= \lim_{t\to 1^-} \sin^{-1}(x)\Big|_0^t \\
&= \lim_{t\to 1^-} \sin^{-1}(t) - \sin^{-1}(0) \\
&= \pi/2 - 0 \\
&= \pi/2
\end{aligned}$$

You can see the process is very much like the previous integrals.

Example 7

Find $\displaystyle\int_1^2 \frac{dx}{x\ln x}$

We'll start by using a u-substitution. Let $u = \ln x$, which gives us $du = \dfrac{1}{x}\,dx$. Since the integral has bounds, we'll do the change of bounds now.

$$u(1) = \ln(1) = 0$$

$$u(2) = \ln(2)$$

Let's get started with the integration.

$$\begin{aligned}
\int_1^2 \frac{dx}{x\ln x} &= \int_0^{\ln 2} \frac{1}{u}\,du \\
&= \lim_{t\to 0^+} \int_t^{\ln 2} \frac{1}{u}\,du \\
&= \lim_{t\to 0^+} \ln|u|\Big|_t^{\ln 2} \\
&= \lim_{t\to 0^+} \ln|\ln 2| - \ln t \\
&= \ln 2 - (-\infty) \\
&= \infty
\end{aligned}$$

Example 8

Find $\int_{-1}^{2} x^{-2}\,dx$

Suppose you didn't check to see if there were any infinite discontinuities. Let's see what we get.

$$\begin{aligned}
\int_{-1}^{2} x^{-2}\,dx &= -\frac{1}{x}\Big|_{-1}^{2} \\
&= -\frac{1}{2} + \frac{1}{-1} \\
&= -3/2
\end{aligned}$$

However, if you use the appropriate improper integral method, you'll find this is incorrect. You start by finding the discontinuity, which exists at $x = 0$. You then break it up into two separate integrals.

$$\int_{-1}^{2} x^{-2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_{0}^{2} x^{-2}\,dx$$

You then integrate each on its own.

1. Find

$$\int_{-1}^{0} x^{-2}\,dx$$

$$\begin{aligned}\int_{-1}^{0} x^{-2}\,dx &= -\frac{1}{x}\bigg|_{-1}^{0}\\ &= \lim_{t\to 0^-} -\frac{1}{x}\bigg|_{-1}^{t}\\ &= \lim_{t\to 0^-} -\frac{1}{t}+\frac{1}{-1}\\ &= \infty\end{aligned}$$

2. Find

$$\int_{0}^{2} x^{-2}\,dx$$

$$\begin{aligned}\int_{0}^{2} x^{-2}\,dx &= \lim_{t\to 0^+} -\frac{1}{x}\bigg|_{t}^{2}\\ &= \lim_{t\to 0^+} -\frac{1}{2}+\frac{1}{t}\\ &= \infty\end{aligned}$$

This shows

$$\int_{-1}^{2} x^{-2}\,dx = \infty$$

which is not $-3/2$.

Now suppose you wanted to integrate

$$\int_{1}^{\infty} \frac{dx}{x+e^{3x}}$$

You'll find that is has no anti-derivative. So we can't evaluate this in a closed form (i.e., we would have to approximate). The next question is, does this converge to a number? In other words, does the integral exist? If you think of the integral as area, is the area finite?

Definition 3: Comparison Test

Suppose f and g are continuous functions where $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then $\displaystyle\int_a^\infty g(x)$ is also convergent.
2. If $\displaystyle\int_a^\infty g(x)\,dx$ diverges, then $\displaystyle\int_a^\infty f(x)\,dx$ also diverges.

The Comparison Test is also valid for improper integrals with infinite discontinuities at the endpoints.

Let's get back to

$$\int_1^\infty \frac{dx}{x+e^{3x}}$$

The downside to the comparison test is you have to some idea if it's going to converge or diverge. This helps figure out if you're suppose to find a nice divergent function or a nice convergent function. Identifying convergent integrals just takes time.

I'm going to guess this integral converges. So I need to find a nice function that's bigger than $\frac{1}{x+e^{3x}}$, whose integral converges. Note that if I can make the denominator "smaller," it makes the whole function larger.

$$x+e^{3x} \geq e^{3x} \text{ when } x \geq 1$$

Therefore,

$$\frac{1}{x+e^{3x}} \leq \frac{1}{e^{3x}}$$

So if I can show $\int_1^\infty \frac{1}{e^{3x}}\,dx$ converges, and show $\frac{1}{x+e^{3x}} \geq 0$ for all $x \geq 1$ (which I don't think needs explaining), then we're good.

$$\begin{aligned}\int_1^\infty e^{-3x}\,dx &= \lim_{t\to\infty}\int_1^t e^{-3x}\,dx \\ &= \lim_{t\to\infty} -\frac{1}{3}e^{-3x}\Big|_1^t \\ &= \lim_{t\to\infty} -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{-3} \\ &= 0 + \frac{1}{3}e^{-3}\end{aligned}$$

Since $\frac{1}{e^{3x}} \geq \frac{1}{x+e^{3x}} \geq 0$ and $\int_1^\infty \frac{1}{e^{3x}}\,dx$ converges, then

$$\int_1^\infty \frac{1}{x+e^{3x}}\,dx \text{ converges}$$