Math 230

Calculus II

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Approximate Integration

Some anti-derivatives are difficult to impossible to find.

For example,

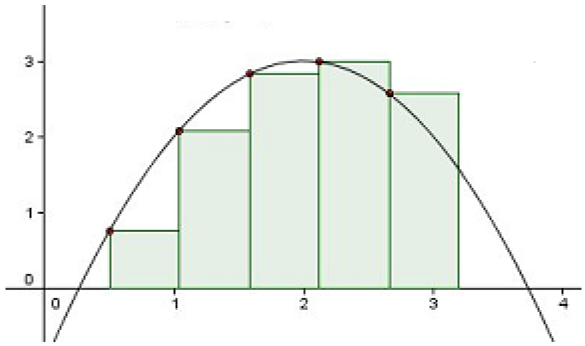
$$\int_0^1 e^{x^2}\,dx \text{ or } \int_{-1}^1 \sqrt{1+x^3}\,dx$$

We came across this situation back in calculus I when we introduced the concept of a definite integral. Since we hadn't talked about the Fundamental Theorem of Calculus, we could only evaluate the area (integrals) by estimating.

We broke the area beneath the curve into a finite number of rectangles. We found the area of each of these rectangles, added them up, and there you go. This was our estimated of the definite integral.

We did this by something like,

Definition 1: Left-Hand Endpoints



Recall that $\Delta x = \dfrac{b-a}{n}$ where n is the number of rectangles.

The area is

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

$$\int_a^b f(x)\,dx = \Delta x\,[f(x_0) + f(x_1) + f(x_2) + \ldots + f(x_{n-1})]$$

where $x_i = a + i\Delta x$.

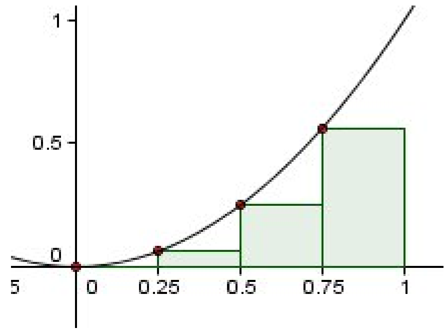
Example 1

Let's estimate $\int_0^1 x^2\,dx$ using left-hand endpoints with $n = 4$ rectangles.

1. Find Δx.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

2. Let's go ahead and draw it.



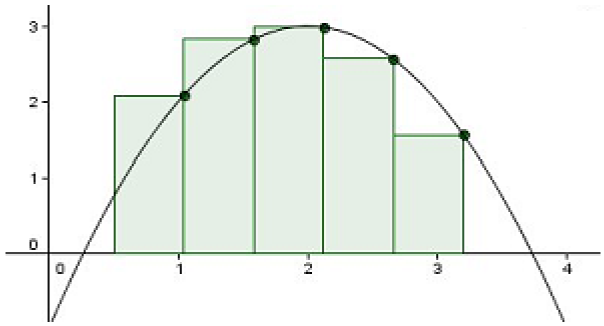
3. The left hand endpoints are

$$x_0 = 0, x_1 = 0.25, x_2 = 0.5, \text{ and } x_3 = 0.75$$

4. The area is

$$\begin{aligned} A &\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &\approx \frac{1}{4}(0^2 + 0.25^2 + 0.5^2 + 0.75^2) \\ &\approx 0.21875 \end{aligned}$$

Definition 2: Right-Hand Endpoints



Recall that $\Delta x = \dfrac{b-a}{n}$ where n is the number of rectangles.

The area is

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \Delta x\,[f(x_1) + f(x_2) + f(x_3) + ... + f(x_n)]$$

where $x_i = a + i\Delta x$.

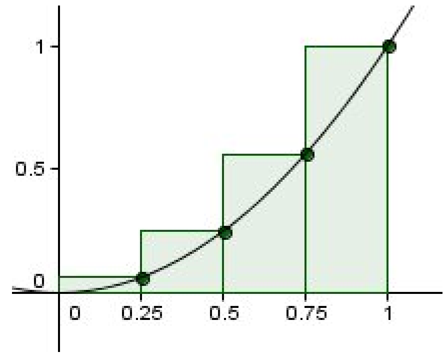
Example 2

Let's estimate $\displaystyle\int_0^1 x^2\,dx$ using right-hand endpoints with $n = 4$ rectangles.

1. Find Δx.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

2. Let's go ahead and draw it.



3. The left hand endpoints are

$$x = 0.25, x = 0.5, x = 0.75, \text{ and } x = 1$$

4. The area is

$$\begin{aligned} A &\approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &\approx \frac{1}{4}(0.25^2 + 0.5^2 + 0.75^2 + 1^2) \\ &\approx 0.46875 \end{aligned}$$

We now have three additional methods to estimate the definite integral. These are useful because they can estimate the area far better than the previous two methods with fewer rectangles or subintervals.

Let's start with the midpoint rule.

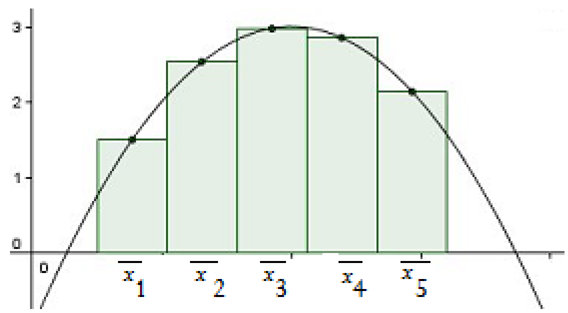
Definition 3: Midpoint Rule

1. You still break up the interval into n rectangles.

2. The width of the rectangle is still

$$\Delta x = \frac{b-a}{n}$$

3. The difference is the height. The height is determined by using the x-value that is the midpoint between the left and right hand endpoints.



4. The area is

$$\int_a^b f(x)\,dx = f(\overline{x_1}) \cdot \Delta x + f(\overline{x_2}) \cdot \Delta x + f(\overline{x_3}) \cdot \Delta x + ... + f(\overline{x_n}) \cdot \Delta x$$

or

$$\int_a^b f(x)\,dx = \Delta x \cdot [f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + ... + f(\overline{x_n})]$$

where $\overline{x_i} = \dfrac{x_i + x_{i-1}}{2}$

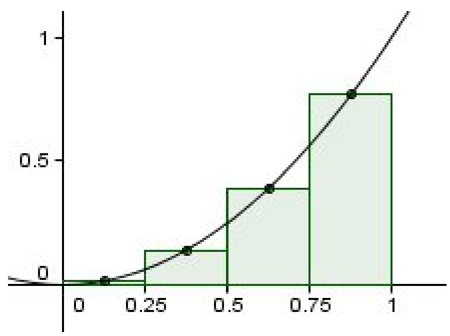
Example 3

Use the Midpoint Rule to estimate $\int_0^1 x^2\,dx$ with $n = 4$ rectangles. We've already done this using the left and right hand endpoints. let's see how much better the estimate is.

1. Find Δx.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

2. Let's go ahead and draw it.



3. The midpoints are

$$\overline{x_1} = \frac{0.25+0}{2} = 0.125$$
$$\overline{x_2} = \frac{0.5+0.25}{2} = 0.375$$
$$\overline{x_3} = \frac{0.75+0.5}{2} = 0.625$$
$$\overline{x_4} = \frac{1+0.75}{2} = 0.875$$

4. The area is

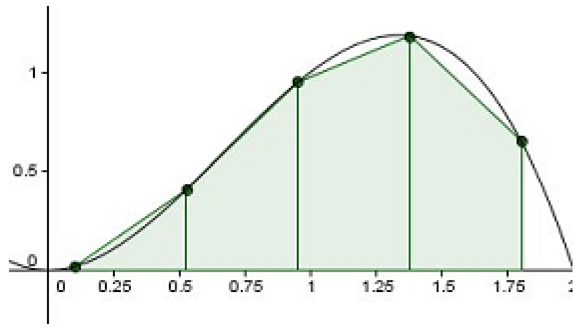
$$\begin{aligned} A &\approx f(\overline{x_1})\Delta x + f(\overline{x_2})\Delta x + f(\overline{x_3})\Delta x + f(\overline{x_4})\Delta x \\ &\approx \frac{1}{4}(.125^2 + 0.375^2 + 0.625^2 + 0.875^2) \\ &\approx 0.3281 \end{aligned}$$

This isn't a bad estimate considering

$$\int_0^1 f(x) \; = \frac{1}{3} = 0.333333...$$

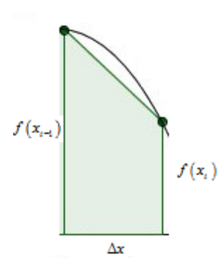
Definition 4: Trapezoidal Rule

After we divide the interval into n segments, we instead make trapezoids instead of rectangles. An example would look something like this,



Notice the trapezoids instead of rectangles?

1. $\Delta x = \dfrac{b-a}{n}$
2. But instead of finding the area of a rectangle, we need to find the area of a trapezoid.



The area this trapezoid is

$$A = \frac{\Delta x}{2}\left[f(x_i) + f(x_{i-1})\right]$$

3. Now we just add up the n trapezoids and we're in business.
4. It would look something like,

$$A \approx \frac{\Delta x}{2}(f(x_0) + f(x_1)) + \frac{\Delta x}{2}(f(x_1) + f(x_2)) + \frac{\Delta x}{2}(f(x_2) + f(x_3)) +$$

$$... + \frac{\Delta x}{2}(f(x_{n-2}) + f(x_{n-1})) + \frac{\Delta x}{2}(f(x_{n-1}) + f(x_n))$$

$$A \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$$

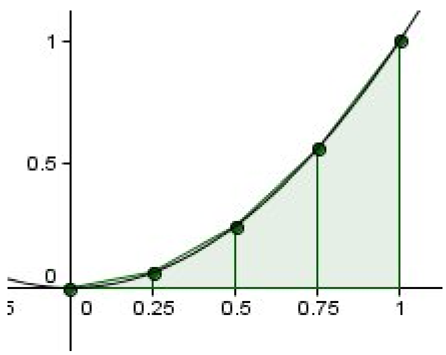
Example 4

Using the Trapezoidal Method estimate $\int_0^1 x^2\,dx$ using $n = 4$ trapezoids.

1. Find Δx

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

2. Draw a picture



3. Find x_i

$$x_0 = 0$$

$$x_1 = 0.25$$

$$x_2 = 0.5$$

$$x_3 = 0.75$$

$$x_4 = 1$$

4. The estimated area is

$$\begin{aligned} A &\approx \frac{1/4}{2}\left[f(0)+2f(0.25)+2f(.5)+2f(.75)+f(1)\right] \\ &\approx \frac{1}{8}\left[0^2+2(0.25)^2+2(0.5)^2+2(0.75)^2+1^2\right] \\ &\approx 0.34375 \end{aligned}$$

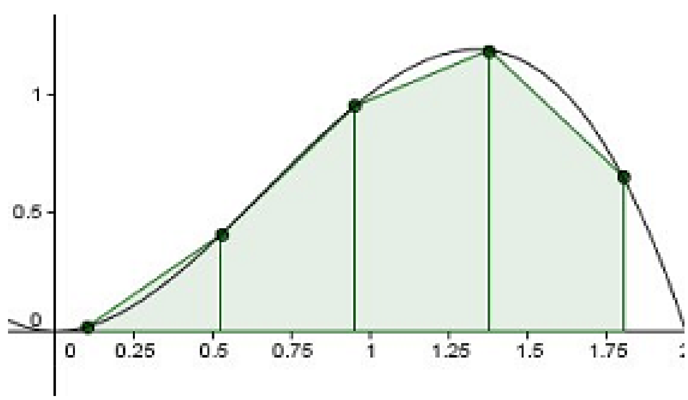
So far we have the following estimates:

Left-Hand:	0.21875	with error	0.114583
Right Hand:	0.46875	with error	0.135417
Midpoint:	0.3281	with error	0.005233
Trapezoid:	0.34375	with error	0.010417

The best estimate is 0.3281 from the Midpoint Rule. In fact, of the four listed above, the Midpoint Rule will give the better approximation. But none are as good as our final method.

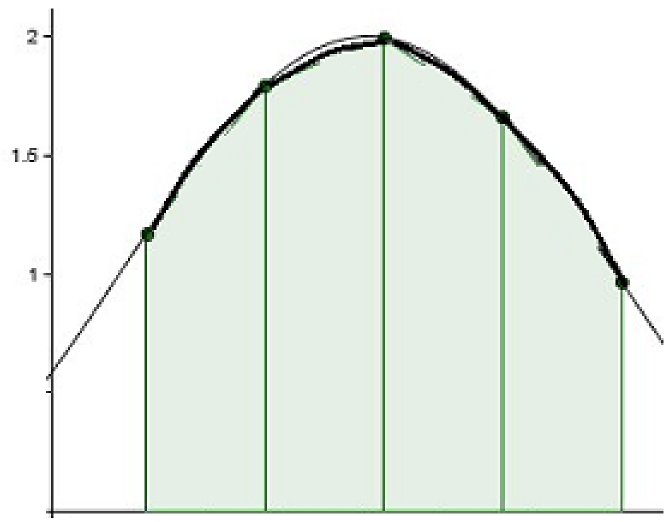
Simpson's Rule is our last approximation method. It is by far the best of the five approximation methods. It is, however, the most complicated. But don't worry. I'll just give you the formula. The proof behind it is a bit daunting. It can be found in almost any calculus II book.

Look back at the trapezoid method,



It seems like it had the right idea. We try to connect the points on top of a function with a straight line (which is how we form a trapezoid instead of the rectangle), and it gave a decent estimate.

But why wasn't it a GREAT estimate? It's because the graph doesn't connect those points by a straight line. They're connected by a **curve**.



I know it's not the greatest drawing, but it's also difficult to distinguish between the approximation region and the real area. Since the top is now approximated by a curve, it fits the area too well to really see a difference. Here's the formula for Simpson's Method.

Definition 5: Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots \right.$$
$$\left. +2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where n is **EVEN** and $\Delta = \dfrac{b-a}{n}$.

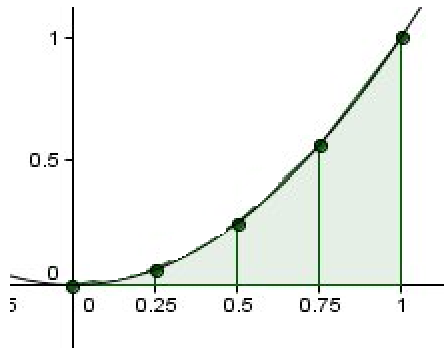
Example 5

Using Simpson's Method, estimate $\displaystyle\int_0^1 x^2\,dx$ with $n = 4$ regions.

1. Find Δx

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}$$

2. Here's where we draw a picture. You probably won't see a difference between the estimated and actual area.



3. What is x_i

$$x_0 = 0$$

$$x_1 = 0.25$$

$$x_2 = 0.5$$

$$x_3 = 0.75$$

$$x_4 = 1$$

4. Use Simpson's Formula

$$\begin{aligned} A &\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &\approx \frac{1/4}{3}\left[0^2 + 4(0.25)^2 + 2(0.5)^2 + 4(0.75)^2 + 1^2\right] \\ &\approx 0.333333 \end{aligned}$$

Look at that! Rounding to 6 decimals and only $n = 4$ regions, we get an extremely good estimate. We can't even tell the difference since we know the true area is $\frac{1}{3}$.

Example 6: Example using all five methods

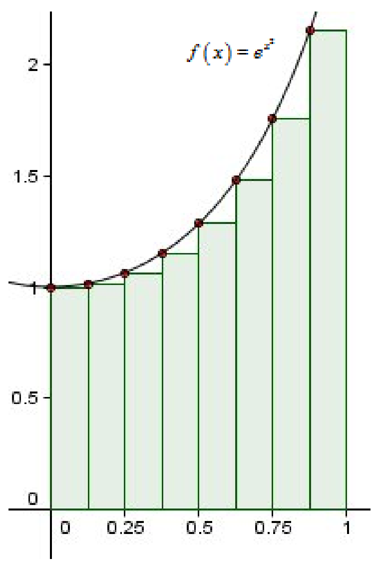
Using all five methods and $n = 8$, estimate $\int_0^1 e^{x^2}\,dx$

1. Left Endpoint Method

   (a) Find Δx.

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

   (b) Let's go ahead and draw it.



   (c) The left hand endpoints are

$$x_0 = 0, x_1 = 0.125, x_2 = 0.25, x_3 = 0.375, x_4 = 0.5$$

$$x_5 = 0.625, x_6 = 0.75, x_7 = 0.875$$

(d) The area is

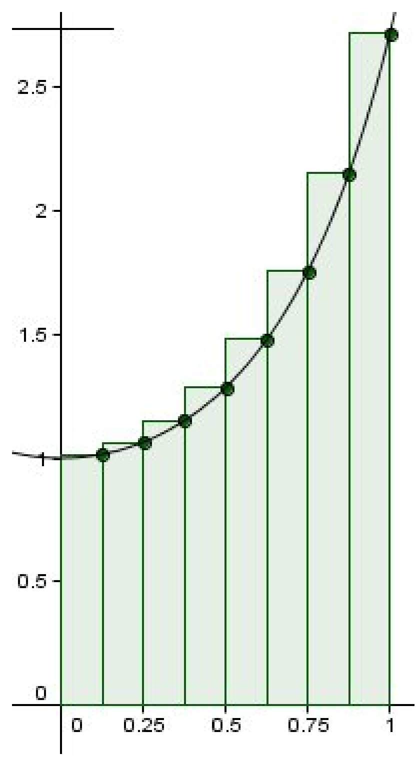
$$
\begin{aligned}
A &\approx \Delta x\left[f(x_0)+f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)\right] \\
&\approx \frac{1}{8}(e^{0^2}+e^{0.125^2}+e^{0.25^2}+e^{0.375^2}+e^{0.5^2}+e^{0.625^2}+e^{0.75^2}+e^{0.875^2}) \\
&\approx 1.3623
\end{aligned}
$$

2. Right Endpoint Method

(a) Find Δx.

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

(b) Let's go ahead and draw it.



(c) The right hand endpoints are

$$x_1 = 0.125, x_2 = 0.25, x_3 = 0.375, x_4 = 0.5$$

$$x_5 = 0.625, x_6 = 0.75, x_7 = 0.875, x_8 = 1$$

(d) The area is

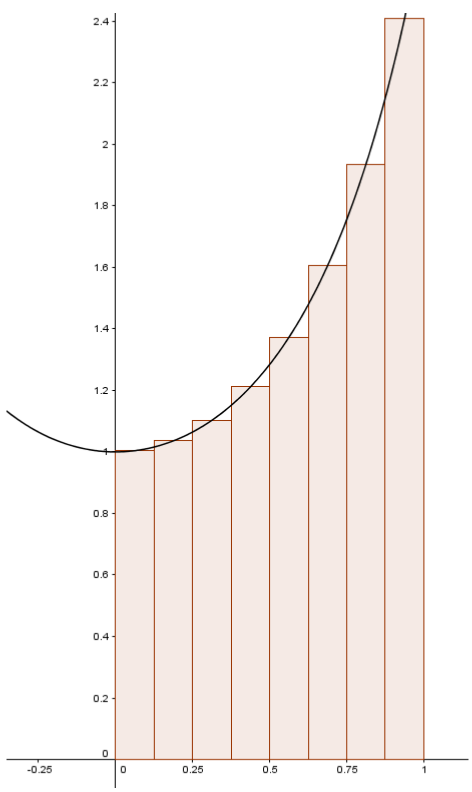
$$\begin{aligned} A &\approx \Delta x \left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7) + f(x_8) \right] \\ &\approx \frac{1}{8}(e^{0.125^2} + e^{0.25^2} + e^{0.375^2} + e^{0.5^2} + e^{0.625^2} + e^{0.75^2} + e^{0.875^2} + e^{1^2}) \\ &\approx 1.5771 \end{aligned}$$

3. Midpoint Method

(a) Find Δx.

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

(b) Let's go ahead and draw it.



(c) The midpoints are

$$\overline{x_1} = 0.0625, \overline{x_2} = 0.1875, \overline{x_3} = 0.3125, \overline{x_4} = 0.4375$$

$$\overline{x_5} = 0.5625, \overline{x_6} = 0.6875, \overline{x_7} = 0.8125, \overline{x_8} = 0.9375$$

(d) The area is

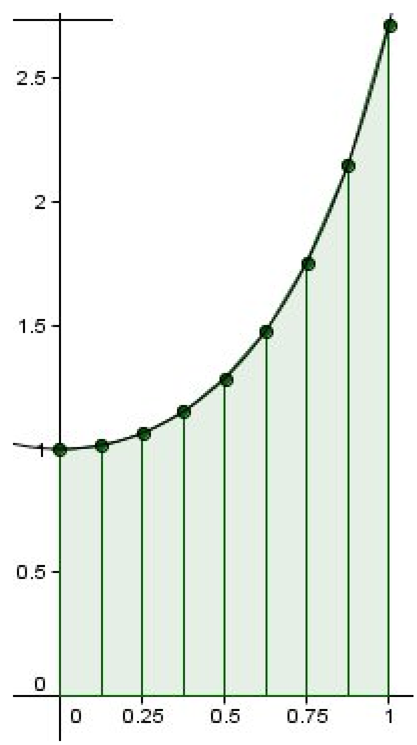
$$\begin{aligned} A &\approx \Delta x \left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + f(\overline{x_4}) + f(\overline{x_5}) + f(\overline{x_6}) + f(\overline{x_7}) + f(\overline{x_8}) \right] \\ &\approx \frac{1}{8}(e^{0.0625^2} + e^{0.1875^2} + e^{0.3125^2} + e^{0.4375^2} + e^{0.5625^2} + e^{0.6875^2} + e^{0.8125^2} + e^{0.9375^2}) \\ &\approx 1.4591 \end{aligned}$$

4. Trapezoidal Method

(a) Find Δx.

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

(b) Let's go ahead and draw it.



(c) The right hand endpoints are

$$x_0 = 0, x_1 = 0.125, x_2 = 0.25, x_3 = 0.375, x_4 = 0.5$$

$$x_5 = 0.625, x_6 = 0.75, x_7 = 0.875, x_8 = 1$$

(d) The area is

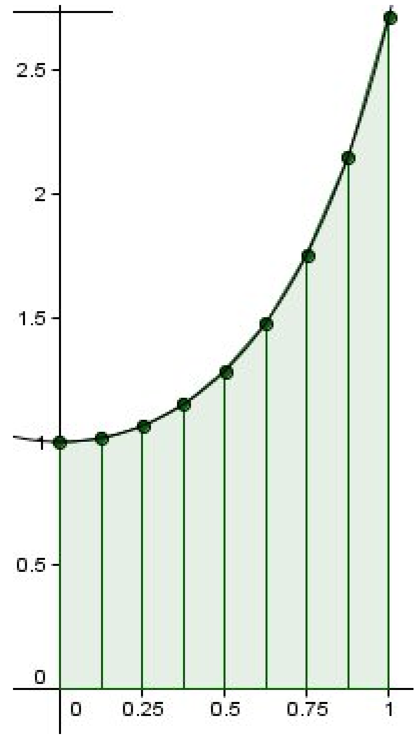
$$\begin{aligned} A &\approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) + 2f(x_7) + f(x_8)\right] \\ &\approx \frac{1/8}{2}(e^{0^2} + 2e^{0.125^2} + 2e^{0.25^2} + 2e^{0.375^2} + 2e^{0.5^2} + 2e^{0.625^2} + 2e^{0.75^2} + 2e^{0.875^2} + e^{1^2}) \\ &\approx 1.4697 \end{aligned}$$

5. Simpson's Method

(a) Find Δx.

$$\Delta x = \frac{1-0}{8} = \frac{1}{8}$$

(b) Let's go ahead and draw it.



(c) The right hand endpoints are

$$x_0 = 0, x_1 = 0.125, x_2 = 0.25, x_3 = 0.375, x_4 = 0.5$$
$$x_5 = 0.625, x_6 = 0.75, x_7 = 0.875, x_8 = 1$$

(d) The area is

$$\begin{aligned} A &\approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\right] \\ &\approx \frac{1/8}{3}(e^{0^2} + 4e^{0.125^2} + 2e^{0.25^2} + 4e^{0.375^2} + 2e^{0.5^2} + 4e^{0.625^2} + 2e^{0.75^2} + 4e^{0.875^2} + e^{1^2}) \\ &\approx 1.46272 \end{aligned}$$

Using approximation techniques are great, right? Well, yes and no. What would be better is if we knew how large n had to be so we could get within a certain error. **Error** is defined to be the amount your approximation is off from the exact value.

$$\text{Error } = \text{Exact } - \text{ Approximation}$$

Error Bounds

Definition 6: Error Bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are the error in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example 7

Find the error for the midpoint method from the previous problem.

1. First, we need to bound $|f''(x)|$ on the interval $[0,1]$.

   If $f(x) = e^{x^2}$, then $f'(x) = 2xe^{x^2}$ and $f''(x) = 2e^{x^2}(2x^2+1)$. So $|f''(x)|$ is largest when $x = 1$. Therefore, $K = 2e^{1^2}(2(1)^2+1) = 16.30969$.

2. We now know $|f''(x)| \leq K = 16.30969$.

3. Using the formula from the definition for error bounds, we have

$$|E_M| = \frac{K(b-a)^3}{24n^2} = \frac{16.30969(1-0)^3}{24(8)^2} = 0.010618$$

4. Conclusion? With $n = 8$ subintervals, the midpoint method is at least accurate to within 0.010618.

Example 8

The next natural question is, 'If I want to be accurate to within 0.001, how many subintervals do I need?' Let's answer that with the formula

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

1. First, you need to find K, which we already did in the previous example.

$$K = 16.30969$$

2. If we know $|E_M| \leq 0.001$, we need to solve for n.

$$\begin{aligned}
\frac{K(b-a)^3}{24n^2} &\leq 0.001 \\
\frac{16.30969(1-0)}{24n^2} &\leq 0.001 \\
\frac{1}{n^2} &\leq \frac{24(0.001)}{16.30969} \\
\frac{1}{n^2} &\leq 0.0014715 \\
n^2 &\geq 679.57 \\
n &\geq 26.068
\end{aligned}$$

3. We need at least 27 subintervals to guarentee an accuracy to within 0.001. Can it be done with fewer? Yes. But the error bound formula guarentees us this accuracy. Plus, having more subintervals just means a better approximation.

Definition 7: Error Bounds for Simpson's Rule

Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_S is the error involved with Simpson's Rule, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Example 9

Using the Error Bounds, find the error for Simpson's Rule for the approximation of $\int_0^1 e^{x^2}\,dx$.

1. First, we need to find $f^{(4)}(x)$.

$$f^{(4)}(x) = 4e^{x^2}(4x^4 + 12x^2 + 3)$$

2. To find K, we need to bound $f^{(4)}(x)$ on the interval $[0, 1]$. This will occur when $x = 1$.

$$|f^{(4)}(x)| \leq f^{(4)}(1) = 206.589419$$

3. Now, we plug everything into the error bound formula to find E_S.

$$|E_S| \leq \frac{K(b-a)^5}{180n^4} = \frac{206.589419(1-0)^5}{180(8)^4} = 0.0002804$$

4. Not a bad estimate considering we only used $n = 8$ subintervals. You can have a computer program run this and get an extremely accurate result.