Math 230

Calculus II

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Integration of Rational Functions by Partial Fractions

From algebra, we learned how to find common denominators so we can do something like this,

$$\frac{2}{x+1} + \frac{3}{x-3} = \frac{2(x-3)}{(x+1)(x-3)} + \frac{3(x+1)}{(x+1)(x-3)} = \frac{5x-3}{x^2-2x-3}$$

So why do we need this? We need to know how to do this in the reverse order. If we're given $\frac{5x-3}{x^2-2x-3}$, we need to know that it can be rewritten as $\frac{2}{x+1} + \frac{3}{x-3}$.

Why you ask?

Let's consider the integral $\int \frac{5x-3}{x^2-2x-3}\, dx$.

We don't have a method that can do this. We can't use u-substitution, trig substitution, integration by parts, and there are no powers of trig functions. But what if we wrote the integral as,

$$\int \frac{5x-3}{x^2-2x-3}\, dx = \int \frac{2}{x+1} + \frac{3}{x-3}\, dx$$

Now we can evaluate this,

$$\int \frac{2}{x+1} + \frac{3}{x-3}\, dx = 2\ln|x+1| + 3\ln|x-3| + C$$

The next objective is given a rational function like $\frac{5x-3}{x^2-2x-3}$, how do we break it up?

Method of Partial Fractions

1. Factor the denominator

$$x^2 - 2x - 3 = (x+1)(x-3)$$

2. Next, rewrite the rational function as

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}$$

The numerators are A and B because the denominators are linear factors.

3. Clear denominators by multiplying by $(x+1)(x-3)$

$$5x - 3 = A(x-3) + B(x+1)$$

$$5x - 3 = Ax - 3A + Bx + B$$

$$5x - 3 = (A+B)x + (-3A+B)$$

4. Match the coefficients

$$\begin{aligned} 5 &= A + B \\ -3 &= -3A + B \end{aligned}$$

5. Solve the resulting systems of equations by

   (a) Using the substitution method

   (b) Using the addition method

   (c) Using row reducing with matrices

Solution: $A = 2$ and $B = 3$.

Therefore, $\dfrac{5x-3}{x^2-2x-3} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$

The next methods require the degree of the numerator to be less than the degree of the denominator. If the degree of the numerator is the same or higher, you must do **long division** before proceeding to the following methods.

Case 1: Denominator is a product of distinct linear factors

Suppose you have a rational function, $\dfrac{P(x)}{Q(x)}$, where the degree of $P(x)$ is smaller than $Q(x)$. If the degree of $P(x)$ is greater than or equal to $Q(x)$, you must use long division.

Find the factors of $Q(x)$.

Example 1

Find $\displaystyle\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$

1. Set up the fraction as,

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}$$

2. Multiply through by $(x-1)(x+1)(x+3)$.

$$x^2+4x+1 = A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1)$$

3. Multiply everything out, collect like terms.

$$x^2+4x+1 = Ax^2+4Ax+3A+Bx^2+2Bx-3B+Cx^2-C$$

$$x^2+4x+1 = (A+B+C)x^2+(4A+2B)x+(3A-3B-C)$$

4. Match coefficients

$$\begin{aligned} A+B+C &= 1 \\ 4A+2B &= 4 \\ 3A-3B-C &= 1 \end{aligned}$$

   (a) Add equations (1) and (3) together and you get

$$4A-2B=2$$

   (b) Pair this with equation (2) and solve

$$\begin{aligned} 4A-2B &= 2 \\ 4A+2B &= 4 \end{aligned}$$

   (c) Add the equations together

$$8A=6 \to A=\frac{3}{4}$$

5. Plugging $A=\frac{3}{4}$ into the other equations, we get

$$A=\frac{3}{4}, B=\frac{1}{2}, C=-\frac{1}{4}$$

6. Rewrite the integral

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \int \frac{3/4}{x-1}+\frac{1/2}{x+1}-\frac{1/4}{x+3}\,dx$$

7. Now we can integrate

$$\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx = \frac{3}{4}\ln|x-1|+\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln|x+3|+C$$

Case 2: Denominator is a product of repeated linear factors

If $Q(x)$ has a factor of $(x-r)^n$, then you have the following partial fraction breakdown,

$$\frac{A}{x-r}+\frac{B}{(x-r)^2}+\frac{C}{(x-r)^3}+...+\frac{N}{(x-r)^n}$$

Example 2

Find $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

1. Note the numerator has a larger degree than the numerator. You need to do long division. After you finish, you should have

$$\int x+1+\frac{4x}{x^3-x^2-x+1}\,dx$$

2. We know how to integrate $x+1$, so we'll just focus on the fraction. Set up the fraction as,

$$\frac{4x}{x^3-x^2-x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$$

3. Multiply through by $(x-1)^2(x+1)$

$$4x = A(x-1)(x+1)+B(x+1)+C(x-1)^2$$
$$4x = Ax^2-A+Bx+B+Cx^2-2Cx+C$$
$$4x = (A+C)x^2+(B-2C)x+(-A+B+C)$$

4. Match coefficients

$$\begin{aligned} A+C &= 0 \\ B-2C &= 4 \\ -A+B+C &= 0 \end{aligned}$$

5. Solving for A, B, C, we get $A = 1, B = 2, C = -1$.

6. Rewrite the integral

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx = \int x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\, dx$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx = \frac{1}{2}x^2 + x + \ln|x-1| - \frac{1}{x-1} - \ln|x+1| + C$$

Case 3: Denominator has irreducible quadratic factors

If $Q(x)$ in $\dfrac{P(x)}{Q(x)}$ has irreducible factors, the partial fraction breakdown will have the term

$$\frac{Ax + B}{ax^2 + bx + c}$$

where $ax^2 + bx + c$ is the irreducible quadratic.

You can tell if a quadratic is irreducible if $b^2 - 4ac < 0$.

Example 3

Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$

1. Since the degree of the denominator is bigger, we can start by writing

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

2. Multiply through by $x(x^2 + 4)$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x$$

$$2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$$

3. Match the coefficients

$$\begin{aligned} A+B &= 2 \\ C &= -1 \\ 4A &= 1 \end{aligned}$$

4. Solving the system of equations we get

$$A = 1, B = 1, C = -1$$

5. Rewrite the integral

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \frac{1}{x} + \frac{1x - 1}{x^2 + 4}\,dx$$

We really can't do much with $\dfrac{x-1}{x^2+4}$ except to break it up into two separate integrals.

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \frac{1}{x} + \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4}\,dx$$

A quick note:

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

and

$$\int \frac{x}{x^2 + 4}\,dx \text{ requires } u\text{-substitution}$$

6. Final Answer

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \ln\left|\frac{1}{x}\right| + \frac{1}{2}\ln\left|x^2 + 4\right| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Case 4: Denominator has repeated irreducible quadratic factors

If $Q(x)$ in $\dfrac{P(x)}{Q(x)}$ has a repeated irreducible factor $(ax^2+bx+c)^n$, the partial fraction breakdown will have the following terms

$$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\frac{Ex+F}{(ax^2+bx+c)^3}+\ldots+\frac{}{(ax^2+bx+c)^n}$$

where ax^2+bx+c is the irreducible quadratic.

Example 4

Find $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

1. Write your fraction as

$$\frac{x^2+x+1}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$$

2. Multiply through by $(x^2+1)^2$

$$x^2+x+1=(Ax+B)(x^2+1)+Cx+D$$

$$x^2+x+1=Ax^3+Bx^2+Ax+B+Cx+D$$

$$x^2+x+1=Ax^3+Bx^2+(A+C)x+(B+D)$$

3. Match the coefficients

$$\begin{aligned} A &= 0 \\ B &= 1 \\ A+C &= 1 \\ B+D &= 1 \end{aligned}$$

4. Solving this system, we get

$$A = 0, B = 1, C = 1, D = 0$$

5. Rewrite the integral

$$\int \frac{x^2 + x + 1}{(x^2+1)^2}\, dx = \int \frac{1}{x^2+1} + \frac{x}{(x^2+1)^2}\, dx$$

6. $\displaystyle\int \frac{x}{(x^2+1)^2}\, dx$ requires u-substitution.

$$\int \frac{x}{(x^2+1)^2}\, dx = -\frac{1}{2(x^2+1)} + C$$

7. Final Answer:

$$\int \frac{x^2 + x + 1}{(x^2+1)^2}\, dx = \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C$$