Math 230

Calculus II

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Trigonometric Substitution

Trig substitution reduces integrals of a certain form to integrals of trig functions, which can be easier to deal with. The idea is to match the integral like so

Steps 1: How to Identify the Trig Substitution

If you see	Substitute	Uses the following Identity
$\sqrt{a^2 - x^2}$	$x = a\sin(\theta)$	$1 - \sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2 + x^2}$	$x = a\tan(\theta)$	$1 + \tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2 - a^2}$	$x = a\sec(\theta)$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

I think it's best we just get right into it.

Example 1

Find $\displaystyle\int \frac{dx}{\sqrt{25 + x^2}}$

Since the bottom has the form $a^2 + x^2$, we use the second option.

Let $x = 5\tan(\theta)$, $dx = 5\sec^2(\theta)\ d\theta$. So,

$$\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5\sec^2(\theta)\ d\theta}{\sqrt{25 + 25\tan^2(\theta)}} = \int \frac{5\sec^2(\theta)\ d\theta}{\sqrt{25(1+\tan^2(\theta))}} = \int \frac{5\sec^2(\theta)\ d\theta}{\sqrt{25\sec^2(\theta)}}$$

$$\int \frac{5\sec^2(\theta)\ d\theta}{5\sec(\theta)} = \int \sec(\theta)\ d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$$

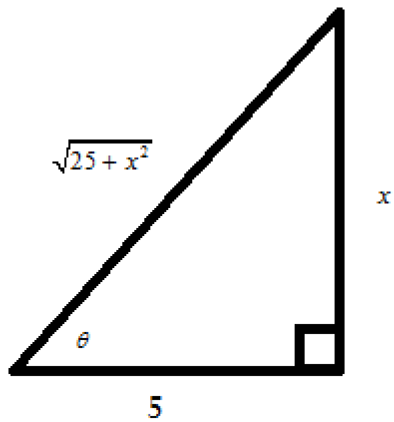
That didn't seem to bad, right? Well, we're not done yet. Remember, this is a substitution problem. The original variable is x. So we need to substitute back. Sometimes it's easy, and sometimes you need a little trig.

1. So what's $\tan(\theta)$

   Since $x = 5\tan(\theta)$, we get $\tan(\theta) = \dfrac{x}{5}$.

2. And $\sec(\theta)$?

   Let's take a look at a right triangle with angle θ.



From here, we see that $\sec(\theta) = \dfrac{\sqrt{25+x^2}}{5}$.

Therefore,

$$\int \frac{dx}{\sqrt{25+x^2}} = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C$$

or

$$\int \frac{dx}{\sqrt{25+x^2}} = \ln|\sqrt{25+x^2} + x| - \ln 5 + C$$

Example 2

How about $\displaystyle\int \frac{x\,dx}{\sqrt{25+x^2}}$

You need to be careful. When you're working through a textbook, all the problems in those sections are usually done using the same method. In this case that method would be trig substitution. And it would work. But, this integral can be done using u-substitution.

1. Using Trig Substitution

   Let $x = 5\tan(\theta)$, $dx = 5\sec^2(\theta)\ d\theta$

$$\int \frac{x\ dx}{\sqrt{25+x^2}} = \int \frac{25\tan(\theta)\sec^2(\theta)\ d\theta}{\sqrt{25\sec^2(\theta)}} = \int \frac{25\tan(\theta)\sec^2(\theta)\ d\theta}{5\sec(\theta)} = \int 5\sec(\theta)\tan(\theta)\ d\theta$$

$$5\sec(\theta) + C = 5\frac{\sqrt{25+x^2}}{5} + C = \sqrt{25+x^2} + C$$

   We used the right triangle from above to change the $\sec(\theta)$.

2. Using u-substitution

   Let $u = 25 + x^2$, $du = 2x\ dx \rightarrow \dfrac{du}{5} = x\ dx$

$$\begin{aligned} \int \frac{x\ dx}{\sqrt{25+x^2}} &= \int \frac{1}{2}u^{-1/2}\ du \\ &= u^{1/2} + C \\ &= \sqrt{25+x^2} + C \end{aligned}$$

   So which one is faster?

Example 3

Find $\int x^3\sqrt{9-x^2}$

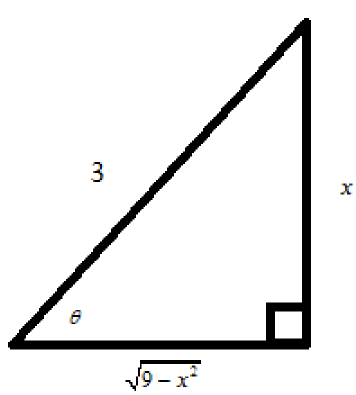
Since the denominator has the form $a^2 - x^2$, we use the substitution $x = 3\sin(\theta)$, and $dx = 3\cos(\theta)\ d\theta$

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= \int 27\sin^3(\theta)\sqrt{9-9\sin^2(\theta)}\cdot(3\cos(\theta))\,d\theta \\
&= \int 27\sin^3(\theta)\sqrt{9\cos^2(\theta)}\cdot 3\cos(\theta)\,d\theta \\
&= 3^5\int \sin^3(\theta)\cos^2(\theta)\,d\theta \\
&= 3^5\int \sin^2(\theta)\cos^2(\theta)\cdot\sin(\theta)\,d\theta \\
&= 3^5\int (1-\cos^2(\theta))\cos^2(\theta)\cdot\sin(\theta)\,d\theta
\end{aligned}$$

Let $u=\cos(\theta)$, and $du=-\sin(\theta)\,d\theta$

$$\begin{aligned}
3^5\int (1-\cos^2(\theta))\cos^2(\theta)\cdot\sin(\theta)\,d\theta &= -3^5\int (1-u^2)u^2\,du \\
&= -3^5\int u^2-u^4\,du \\
&= -3^5\left[\frac{1}{3}u^3-\frac{1}{5}u^5\right] \\
&= -3^5\left[\frac{\cos^3(\theta)}{3}-\frac{\cos^5(\theta)}{5}\right]
\end{aligned}$$

Now we just have to figure out what $\cos(\theta)$ is. To do that, we use the right triangle again. Recall that $x=3\sin(\theta)$



We see that $\cos(\theta) = \dfrac{\sqrt{9-x^2}}{3} = \dfrac{(9-x^2)^{1/2}}{3}$. Now we just replace the $\cos(\theta)$ with this in our integral and we're done!

Note $\cos^3(\theta) = \left(\dfrac{\sqrt{9-x^2}}{3}\right)^3 = \dfrac{(9-x^2)^{3/2}}{3^3}$.

$$\begin{aligned}
\int x^3\sqrt{9-x^2}\,dx &= -3^5\left[\frac{1}{3^4}(9-x^2)^{3/2} - \frac{1}{5\cdot 3^5}(9-x^2)^{5/2}\right] + C \\
&= -\frac{(9-x^2)^{3/2}}{5}\left[15-(9-x^2)\right] + C \\
&= -\frac{(9-x^2)^{3/2}}{5}\left(x^2+6\right) + C
\end{aligned}$$

The last few steps weren't necessary, but it doesn't hurt to practice simplifying.

Example 4

Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2-36}}$

Let $x = 6\sec(\theta)$, and $dx = 6\sec(\theta)\tan(\theta)d\theta$.

Before setting up the integral, we can substitute and simplify $\sqrt{x^2-36}$ right now.

$$\sqrt{x^2-36} = \sqrt{36(\sec^2(\theta)-9)} = 6\tan(\theta)$$

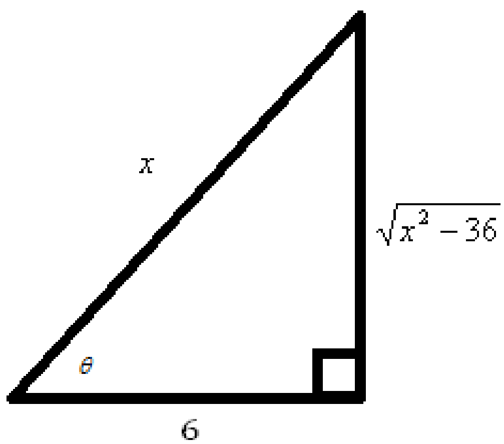
and

$$x^2 = 36\sec^2(\theta)$$

Now we substitute,

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-36}} &= \int \frac{1}{36\sec^2(\theta)\cdot 6\tan(\theta)}\cdot 6\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{1}{36\sec(\theta)}\,d\theta \\ &= \int \frac{1}{36}\cos(\theta)\,d\theta \\ &= \frac{1}{36}\sin(\theta)+C\end{aligned}$$

We need to use the right triangle to rewrite $\sin(\theta)$ in terms of x.
If $x = 6\sec(\theta)$, then $\frac{x}{6} = \sec(\theta) \rightarrow \cos(\theta) = \frac{6}{x}$



From here you can see $\sin(\theta) = \frac{\sqrt{x^2-36}}{x}$
Therefore, $\int \frac{1}{x^2\sqrt{x^2-36}} = \frac{1}{36}\cdot\frac{\sqrt{x^2-36}}{x} + C$

Example 5

Find $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Since the denominator has the form $a^2 - x^2$, we use the following substitution,

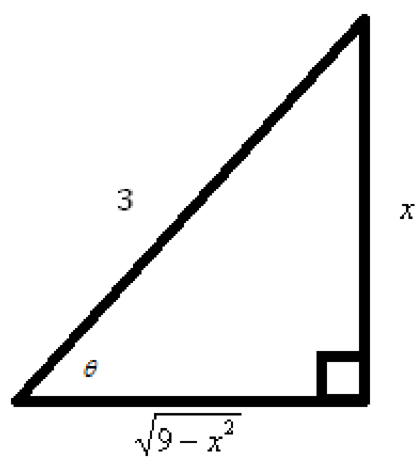
$$x = 3\sin(\theta)\ ,\ dx = 3\cos(\theta)$$

To help simplify before we start the integral,

$$\sqrt{9-x^2} = \sqrt{9-9\sin^2(\theta)} = 3\cos(\theta)$$

$$\begin{aligned}
\int \frac{x}{\sqrt{9-x^2}}\,dx &= \int \frac{9\sin^2(\theta)}{3\cos(\theta)}\cdot 3\cos(\theta)\;d\theta \\
&= \int 9\sin^2(\theta)\;d\theta \\
&= 9\int \frac{1}{2}\left(1-\cos(2\theta)\right)\;d\theta \\
&= \frac{9}{2}\int 1-\cos(\theta)\;d\theta \\
&= \frac{9}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C
\end{aligned}$$

Now we use the right triangle,



Note that we can't use $\sin(2\theta)$, since the angle is just a single θ. To deal with this, we use the trig identity

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\theta - \frac{9}{2}\sin(\theta)\cos(\theta)$$

Using the triangle, we have $\theta = \sin^{-1}\left(\frac{x}{3}\right)$, $\sin(\theta) = \frac{x}{3}$, and $\cos(\theta) = \frac{\sqrt{9-x^2}}{3}$

Final Answer:

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{9}\right] + C$$

Example 6

Find $\int \frac{x}{\sqrt{x^2+4x+5}}\,dx$

The problem here is what's in the square root doesn't follow one of our three forms, which are x^2-a^2, x^2+a^2, or a^2-x^2. So we need to complete the square on x^2+4x+5 so it has one of these forms.

$$x^2+4x+5 = \left(x^2+4x+?\right) -? + 5$$

The missing number ?, is ? $= \left(\frac{-b}{2a}\right)^2 = \left(\frac{-4}{2(1)}\right)^2 = (-2)^2 = 4.$

$$x^2+4x+5 = \left(x^2+4x+4\right) - 4 + 5$$

$$x^2+4x+5 = (x+2)^2 + 1$$

Now we have one of the correct forms. It has the form x^2+a^2, well sorta. But here's what we have now.

$$\int \frac{x}{\sqrt{x^2+4x+5}}\,dx = \int \frac{x}{\sqrt{(x+2)^2+1}}\,dx$$

We need to substitute.

1. Let $u = x+2$ and $x = u-2$

2. $du = dx$

3. Substitute

$$\int \frac{x}{\sqrt{(x+2)^2+1}}\,dx = \int \frac{u-2}{\sqrt{u^2+1}}\,du$$

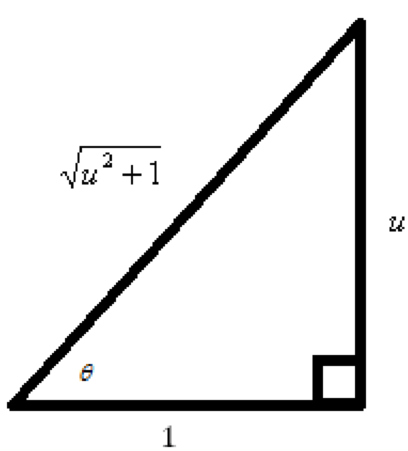
4. Now we do the trig substitution

   Let $x = 1\tan(\theta)$, $du = \sec^2(\theta)\,d\theta$.

$$\sqrt{u^2+1} = \sqrt{\tan^2(\theta)+1} = \sqrt{\sec^2(\theta)} = \sec(\theta)$$

$$\begin{aligned}
\int \frac{x}{\sqrt{(x+2)^2}}\,dx &= \int \frac{u-2}{\sqrt{u^2+1}}\,du \\
&= \int \frac{\tan(\theta)-2}{\sec(\theta)}\cdot \sec^2(\theta)\,d\theta \\
&= \int \sec(\theta)\tan(\theta) - 2\sec(\theta)\,d\theta \\
&= \sec(\theta) - 2\int \sec(\theta) \\
&= \sec(\theta) - 2\ln|\sec(\theta)+\tan(\theta)| + C
\end{aligned}$$

   Now we use the right triangle.



From here we get $\sec(\theta) = \dfrac{\sqrt{u^2+1}}{1}$ and $\tan(\theta) = u$

$$\sec(\theta) - 2\ln|\sec(\theta) + \tan(\theta)| + C = \sqrt{u^2+1} - 2\ln\left|\sqrt{u^2+1} + u\right| + C$$

$$\int \frac{x}{\sqrt{x^2+4x+5}} = \sqrt{x^2+4x+5} - 2\ln\left|\sqrt{x^2+4x+5} + (x+2)\right| + C$$