Math 230

Calculus II

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Trigonometric Integrals

Suppose you had to find the integral

$$\int \cos(x)\sin^5(x)\ dx$$

What would you do?

We use u-substitution. Let $u = \sin(x)$, then $du = \cos(x)\ dx$

$$\begin{aligned} \int \cos(x)\sin^5(x)\ dx &= \int u^5\ du \\ &= \frac{1}{6}u^6 + C \\ &= \frac{1}{6}\sin^6(x) + C \end{aligned}$$

So what if I asked you to evaluate

$$\int \sin^5(x)\ dx$$

To use substitution, you need the derivative of u, which was $\cos(x)$. In this integral, there is no $\cos(x)$. So we need to rewrite $\sin^5(x)$... somehow

Since we need to have a factor of $\cos(x)$ show up, we use a trig identity that allows us to convert $\sin^2(x)$ to $1-\cos^2(x)$.

$$\begin{aligned} \sin^5(x) &= \sin^4(x)\sin(x) \\ &= (1-\cos^2(x))^2\sin(x) \end{aligned}$$

Our new integral is

$$\int \sin^5(x)\,dx = \int (1-\cos^2(x))^2 \sin(x)\,dx$$

1. Let $u = \cos(x)$
2. $du = -\sin(x)\,dx$
3. Substitute

$$\begin{aligned}
\int (1-\cos^2(x))^2 \sin(x)\,dx &= \int (1-u^2)^2\,(-du) \\
&= -\int 1 - 2u^2 + u^4\,du \\
&= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) \\
&= -\left(\cos(x) - \frac{2}{3}\cos^3(x) + \frac{1}{5}\cos^5(x)\right) \\
&= -\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x)
\end{aligned}$$

Example 1

Find $\int \sin^6(x)\cos^3(x)\,dx$

Rewrite

$$\sin^6(x)\cos^3(x) = \sin^6(x)\cos^2(x)\cdot\cos(x)$$

$$\begin{aligned}
\int \sin^6(x)\cos^3(x)\,dx &= \int \sin^6(x)\cos^2(x)\cdot\cos(x)\,dx \\
&= \int \sin^6(x)(1-\sin^2(x))\cdot\cos(x)\,dx
\end{aligned}$$

Let $u = \sin(x)$ and $du = \cos(x)\,dx$. We now substitute

$$\begin{aligned}\int \sin^6(x)(1-\sin^2(x))\cdot\cos(x)\,dx &= \int u^6(1-u^2)\,du \\ &= \int u^6 - u^8\,du \\ &= \frac{1}{7}u^7 - \frac{1}{9}u^9 + C \\ &= \frac{1}{7}\sin^7(x) - \frac{1}{9}\sin^9(x) + C\end{aligned}$$

Steps 1: Strategy for Evaluating $\int \sin^m(x)\cos^n(x)\,dx$

1. If n is odd (the power on $\cos(x)$), save one $\cos(x)$ factor and use

$$\cos^2(x) = 1 - \sin^2(x)$$

to change all the even powers of $\cos(x)$.

Then use u-substitution. Let $u = \sin(x)$ and $du = \cos(x)\,dx$

2. If m is odd (the power on $\sin(x)$), save one $\sin(x)$ factor and use

$$\sin^x(x) = 1 - \cos^2(x)$$

to change all the even powers of $\sin(x)$.

Then use u-substitution. Let $u = \cos(x)$ and $du = -\sin(x)$

3. If both are odd, you choose (1) or (2).

4. If both are even, you can try using the half-angle identities.

$$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$$
$$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

Example 2

Find $\displaystyle\int_0^{\pi} \sin^2(x)\cos^2(x)\,dx$

While we integrate, I'm going to drop the bounds.

$$
\begin{aligned}
\int \sin^2(x)\cos^2(x)\,dx &= \int \frac{1}{2}(1-\cos(2x))\cdot\frac{1}{2}(1+\cos(2x))\,dx \\
&= \frac{1}{4}\int 1-\cos^2(2x)\,dx \\
&= \frac{1}{4}\int 1-\frac{1}{2}(1+\cos(4x))\,dx \\
&= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx \\
&= \frac{1}{8}\int 1-\cos(4x)\,dx \\
&= \frac{1}{8}\left[x-\frac{1}{4}\sin(4x)\right] \\
&= \frac{1}{8}x-\frac{1}{32}\sin(4x)
\end{aligned}
$$

Therefore,

$$
\int_0^{\pi} \sin^2(x)\cos^2(x)\,dx = \frac{1}{8}x-\frac{1}{32}\sin(4x)\Big|_0^{\pi}
$$

$$
\int_0^{\pi} \sin^2(x)\cos^2(x)\,dx = \frac{1}{8}\pi
$$

Steps 2: Strategy for $\int \tan^m(x)\sec^n(x)\,dx$

1. If n is even, save one factor of $\sec^2(x)$. Write all the other $\sec^2(x)$ as

$$\sec^2(x) = 1 + \tan^2(x)$$

   Then use u-substitution. Let $u = \tan(x)$ and $du = \sec^2(x)$.

2. If m is odd, save one factor of $\sec(x)$ and one $\tan(x)$. Rewrite all the other $\tan^2(x)$ as

$$\tan^2(x) = \sec^2(x) - 1$$

   Then use u-substitution. Let $u = \sec(x)$ and $du = \sec(x)\tan(x)$.

Example 3

Find $\int \tan^6(x)\sec^4(x)\,dx$

The power on $\sec(x)$ is even. So pull one factor of $\sec^2(x)$ out.

$$\begin{aligned}\int \tan^6(x)\sec^4(x)\,dx &= \int \tan^6(x)\sec^2(x)\sec^2(x)\,dx \\ &= \int \tan^6(x)(1+\tan^2(x))\cdot\sec^2(x)\,dx\end{aligned}$$

Let $u = \tan(x)$, $du = \sec^2(x)\ dx$

$$\begin{aligned}
\int \tan^6(x)(1+\tan^2(x))\cdot \sec^2(x)\ dx &= \int u^6(1+u^2)\ du \\
&= \int u^6 + u^8\ du \\
&= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C \\
&= \frac{1}{7}\tan^6(x) + \frac{1}{9}\tan^9(x) + C
\end{aligned}$$

Example 4

Find $\int \sec^9 x \tan^5\ dx$

The power on $\tan(x)$ is odd. So pull out a factor of $\sec(x)$ and $\tan(x)$.

$$\begin{aligned}
\int \sec^9 x \tan^5 x\ dx &= \int \sec^8 x \tan^4 x \cdot \sec x \tan x\ dx \\
&= \int \sec^8 x (\tan^2 x)^2 \sec(x)\tan(x)\ dx \\
&= \int \sec^8 x(\sec^2(x)-1)^2 \sec(x)\tan(x)\ dx
\end{aligned}$$

Let $u = \sec(x)$ and $du = \sec(x)\tan(x)$

$$\begin{aligned}
\int \sec^8 x \left(\sec^2(x)-1\right)^2 \sec(x)\tan(x)\ dx &= \int u^8(u^2-1)^2\ du \\
&= \int u^8(u^4-2u^2+1)\ du \\
&= \int u^{12} - 2u^{10} + u^8\ du \\
&= \frac{1}{13}u^{13} - \frac{2}{11}u^{11} + \frac{1}{9}u^9 + C \\
&= \frac{1}{13}\sec^{13} x - \frac{2}{11}\sec^{11} x + \frac{1}{9}\sec^9 x + C
\end{aligned}$$

Example 5

Show $\int \sec(x)\, dx = \ln|\sec(x) + \tan(x)| + C$

Unfortunately, this integral requires a bit of a trick.

$$\sec(x) = \sec(x) \cdot \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}$$

$$\sec(x) = \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}$$

Let $u = \sec(x) + \tan(x)$ and $du = \sec(x)\tan(x) + \sec^2(x)$. This is just a normal u-substitution.

$$\begin{aligned}
\int \sec(x)\, dx &= \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}\, dx \\
&= \int \frac{1}{u}\, du \\
&= \ln|u| + C \\
&= \ln|\sec(x) + \tan(x)| + C
\end{aligned}$$

Example 6

Find $\int \tan^3 x\, dx$

The problem with this one is that there are no powers of $\sec(x)$. When this happens, try pulling out some $\tan^2(x)$.

$$\tan^3(x) = \tan(x) \cdot \tan^2(x) = \tan(x)(\sec^2 x - 1)$$

SEE! We now have a $\sec(x)$.

$$\begin{aligned}\int \tan^3 x\,dx &= \int \tan(x)\cdot\tan^2(x)\,dx\\ &= \int \tan(x)\left(\sec^2(x)-1\right)\,dx\\ &= \int \tan(x)\sec^2(x)-\tan(x)\,dx\end{aligned}$$

Both of these terms are done using u-substitution. Let's start with the first one.

1. $\int \tan(x)\sec^2(x)\,dx$

   Let $u=\sec(x)$, $du=\sec(x)\tan(x)$

$$\begin{aligned}\int \tan(x)\sec^2(x)\,dx &= \int \sec(x)\cdot\sec(x)\tan(x)\,dx\\ &= \int u\,du\\ &= \frac{1}{2}u^2\\ &= \frac{1}{2}\sec^2(x)+C\end{aligned}$$

   You could have used $u=\tan(x)$ and $du=\sec^2(x)$. If you chose this substitution, you'd get

$$\begin{aligned}\int \tan(x)\sec^2(x)\,dx &= \int u\,du\\ &= \frac{1}{2}u^2+C\\ &= \frac{1}{2}\tan^2(x)+C\end{aligned}$$

   It might seem that our two substitutions gave us different results. They really didn't since $\frac{1}{2}\sec^2(x)$ and $\frac{1}{2}\tan^2(x)$ differ by a constant. Remember that $1+\tan^2(x)=\sec^2(x)$. That's why that $+C$ is important.

2. $\int \tan(x)\ dx$

   Rewrite $\tan(x)$ as $\dfrac{\sin(x)}{\cos(x)}$

   Let $u = \cos(x)$, $du = -\sin(x)\ dx \to -du = \sin(x)\ dx$

$$\begin{aligned}\int \tan(x) &= \int \frac{\sin(x)}{\cos(x)}\ dx \\ &= \int \frac{1}{u}(-du) \\ &= -\ln|u| \\ &= -\ln|\cos(x)| \\ &= \ln|\sec(x)|\end{aligned}$$

Final Answer:

$$\int \tan^3(x)\ dx = \frac{1}{2}\sec^2(x) - \ln|\sec(x)| + C$$

Example 7

Find $\int \sec^3(x)\ dx$

Since this doesn't have any powers of $\tan(x)$, we need another way to approach this. Also, the power of $\sec(x)$ is not even, so we cannot even try the strategy from above.

We do this problem by **Integration by Parts**

$$\begin{aligned} u &= \sec(x) & dv &= \sec^2(x)\ dx \\ du &= \sec(x)\tan(x)\ dx & v &= \tan(x)\end{aligned}$$

$$\begin{aligned}
\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx \\
&= \sec(x)\tan(x) - \int \sec(x)(\sec^2(x)-1)\,dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \int \sec(x)\,dx \\
&= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x)+\tan(x)|
\end{aligned}$$

Notice that the right hand side has $\int \sec^3(x)\,dx$. Let's bring it to the right hand side.

$$\begin{aligned}
\int \sec^3(x)\,dx &= \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x)+\tan(x)| \\
2\int \sec^3(x)\,dx &= \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| \\
\int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x)+\tan(x)|
\end{aligned}$$

Formula 1: Integrals So far!

1. $\int k\ dx = kx + C$
2. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
3. $\int \frac{1}{x}\ dx = \ln|x| + C$
4. $\int \frac{1}{kx+b}\ dx = \frac{1}{k}\ln|kx+b| + C$
5. $\int e^x\ dx = e^x + C$
6. $\int e^{kx}\ dx = \frac{1}{k}e^{kx} + C$
7. $\int a^x\ dx = \frac{a^x}{\ln a} + C$
8. $\int a^{kx}\ dx = \frac{1}{k\ln a}a^{kx} + C$
9. $\int \ln x\ dx = x\ln x - x + C$
10. $\int \cos x\ dx = \sin x + C$
11. $\int \sin x\ dx = -\cos x + C$
12. $\int \sec^2 x\ dx = \tan x + C$
13. $\int \csc^2 x\ dx = -\cot x + C$
14. $\int \sec x \tan x\ dx = \sec x + C$
15. $\int \csc x \cot x\ dx = -\csc x + C$
16. $\int \tan x\ dx = \ln|\sec x| + C$
17. $\int \cot x\ dx = \ln|\sin x| + C$
18. $\int \csc x\ dx = \ln|\csc x - \cot x| + C$
19. $\int \sec x\ dx = \ln|\sec x + \tan x| + C$
20. $\int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
21. $\int \frac{1}{a^2 + x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int -\frac{1}{a^2 + x^2}\ dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2 - 1}}\ dx = \sec^{-1}(x) + C$
24. $\int -\frac{1}{x\sqrt{x^2 - 1}}\ dx = \csc^{-1}(x) + C$