Math 230

Calculus II

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Integration By Parts

If u and v are functions of x, the Product Rule says that

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides:

$$\int \frac{d}{dx}uv\ dx = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ dx$$

$$uv = \int u\ dv + \int v\ du$$

$$\int u\ dv = uv - \int v\ du$$

The left hand $\int u\ dv$ is the integral we're trying to evaluate. Our goal is to choose a u and a dv.

Definition 1: Integration By Parts

$$\int u\ dv = uv - \int v\ du$$

$$\int_a^b u\ dv = uv\Big|_a^b - \int_a^b v\ du$$

Example 1

Find $\displaystyle\int xe^{6x}\ dx$

$$\text{Let } u = x \qquad dv = e^{6x}\,dx$$
$$du = dx \qquad v = \frac{1}{6}e^{6x}$$

Now it's time to use Integration by Parts.

$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ &= \frac{xe^{6x}}{6} - \int \frac{1}{6}e^{6x}\,dx \\ &= \frac{xe^{6x}}{6} - \frac{1}{36}e^{6x} + C\end{aligned}$$

Example 2

Find $\int (x^2+1)\sin(x)\,dx$

$$\text{Let } u = x^2+1 \qquad dv = \sin(x)$$
$$du = 2x\,dx \qquad v = -\cos(x)$$

$$\int (x^2+1)\sin(x)\,dx = -(x^2+1)\cos(x) + \int 2x\cos(x)$$

So now we have to integrate $\int 2x\cos(x)\,dx$, which requires integration by parts again.

$$\text{Let } u = 2x \qquad dv = \cos(x)\,dx$$
$$du = 2\,dx \qquad v = \sin(x)$$

$$\int (x^2+1)\sin(x)\,dx = -(x^2+1)\cos(x) + \left[2x\sin(x) - \int 2\sin(x)\,dx\right]$$

$$\int (x^2+1)\sin(x)\ dx = -(x^2+1)\cos(x) + 2x\sin(x) + 2\cos(x) + C$$

Example 3

Find $\int \ln x\ dx$

$$\text{Let } \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\ dx \qquad v = x$$

$$\begin{aligned} \int \ln x\ dx &= uv - \int v\ du \\ &= x\ln x - \int x\ \frac{1}{x}\ dx \\ &= x\ln x - \int 1\ dx \\ &= x\ln x - x + C \end{aligned}$$

Example 4

Find $\int (\ln x)^2\ dx$

$$u = (\ln x)^2 \qquad dv = dx$$
$$du = 2\ln(x)\cdot\frac{1}{x}\ dx \qquad v = x$$

$$\begin{aligned}\int (\ln x)^2\,dx &= uv - \int v\,du \\ &= x(\ln x)^2 - \int 2\frac{\ln(x)}{x}\cdot x\,dx \\ &= x(\ln x)^2 - \int 2\ln(x)\,dx \\ &= x(\ln x)^2 - 2(x\ln x - x) + C \\ &= x(\ln x)^2 - 2x\ln x + 2x + C\end{aligned}$$

Example 5

Find $\int \sin^{-1}(x)\,dx$

$$\begin{aligned}&\text{Let } u = \sin^{-1}(x) && dv = dx \\ &du = \frac{1}{\sqrt{1-x^2}}\,dx && v = x\end{aligned}$$

$$\int \sin^{-1}(x)\,dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Notice the new integral doesn't require integration by parts. Instead it requires u-substitution.

1. Let $u = 1 - x^2$
2. $du = -2x\,dx \to -\frac{1}{2}\,du = x\,dx$
3. Substitute

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= \int \frac{1}{\sqrt{1-x^2}}\cdot x\,dx \\ &= \int -\frac{1}{2}u^{-1/2}\,du \\ &= -u^{1/2} \\ &= -(1-x^2)^{1/2}\end{aligned}$$

Now let's finish the problem

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) - \left[-(1-x^2)^{1/2}\right]$$

$$\int \sin^{-1}(x)\, dx = x\sin^{-1}(x) + (1-x^2)^{1/2} + C$$

Example 6

Find $\int_0^1 \sin^{-1}(x)\, dx$

I suggest you ignore the bounds for now. Integrate using by parts and evaluate once you're finished integration.

$$\int_0^1 \sin^{-1}(x)\, dx = x\sin^{-1}(x) + (1-x^2)^{1/2}\Big|_0^1$$

$$= \left[1\sin^{-1}(1) + (1-1^2)^{1/2}\right] - \left[0\sin^{-1}(0) + (1-0^2)^{1/2}\right]$$

$$= \pi/2 - 1$$

Example 7

$\int e^{-x}\cos(2x)\, dx$

$$u = e^{-x} \qquad dv = \cos(2x)\, dx$$

$$du = -e^{-x}\, dx \qquad v = \frac{1}{2}\sin(2x)$$

$$\begin{aligned}
\int e^{-x}\cos(2x)\,dx &= uv - \int v\,du \\
&= \frac{1}{2}e^{-x}\sin(2x) - \int -\frac{1}{2}e^{-x}\sin(2x)\,dx \\
&= \frac{1}{2}e^{-x}\sin(2x) + \int \frac{1}{2}e^{-x}\sin(2x)\,dx
\end{aligned}$$

Note that $\int \frac{1}{2}e^{-x}\sin(2x)\,dx$ requires its own integration by parts.

$$u = e^{-x} \qquad dv = \frac{1}{2}\sin(2x)\,dx$$

$$du = -e^{-x}\,dx \qquad v = -\frac{1}{4}\cos(2x)$$

$$\begin{aligned}
\int e^{-x}\sin(2x)\,dx &= uv - \int v\,du \\
&= \frac{1}{4}e^{-x}\cos(2x) - \int \frac{1}{4}e^{-x}\cos(2x)\,dx \\
&= \frac{1}{4}e^{-x}\cos(2x) - \int \frac{1}{4}e^{-x}\cos(2x)\,dx
\end{aligned}$$

So what we have at this point is the following

$$\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{4}e^{-x}\cos(2x) - \int \frac{1}{4}e^{-x}\cos(2x)\,dx$$

Note how they original integral shows up on the right hand side of the equation. We can bring it to the left hand side with the original integral (since they are like terms).

$$\frac{5}{4}\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{4}e^{-x}\cos(2x)$$

Now, multiply both sides by $\frac{4}{5}$ to solve for the original integral.

$$\int e^{-x}\cos(2x)\,dx = \frac{2}{5}e^{-x}\sin(2x) + \frac{1}{5}e^{-x}\cos(2x) + C$$

Example 8

Find $\int \cos(\sqrt{x})\, dx$.

1. This is an example of where a u-substitution would come in handy.

2. Let $u = \sqrt{x}$

3. Then $du = \frac{1}{2\sqrt{x}}\, dx$. Note there is no $\frac{1}{\sqrt{x}}$ anywhere in the integral. Instead, we let

$$\begin{aligned} du &= \frac{1}{2u}\, dx \\ 2u\, du &= dx \end{aligned}$$

4. Now we substitute

$$\begin{aligned} \int \cos(\sqrt{x})\, dx &= \int \cos(u)\, 2u\, du \\ &= \int 2u \cos(u)\, du \end{aligned}$$

5. Now, we have an integral set up for integration by parts. Since we already have u, we will use w and v

Let $w = 2u$ $\qquad$ $dv = \cos(u)\, du$

$dw = 2\, du$ $\qquad$ $v = \sin(u)$

6. Now we start integration by parts

$$\begin{aligned} \int 2u \cos(u)\, du &= wv - \int v\, dw \\ &= 2u \sin(u) - \int 2 \sin(u)\, du \\ &= 2u \sin(u) + 2 \cos(u) + C \end{aligned}$$

7. Now substitute back $u = \sqrt{x}$.

$$\int \cos(\sqrt{x})\,dx = 2(\sqrt{x})\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$$