Math 230

Calculus II

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Indeterminate Forms and L'Hospital's Rule

From calculus I, we used a geometric approach to show

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1$$

We needed to use a geometric approach back then because we didn't know what to do with $\dfrac{0}{0}$ and we couldn't use one of our limit methods (like using conjguates, factoring, etc.).

In general, if you have a limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

and $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$, then the limit **may or may not** exist. That's for us to find out.

Example 1: Indeterminate Form of Type $\frac{0}{0}$

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

because $\lim_{x\to 0}\sin(x)=0$ and $\lim_{x\to 0}x=0$

Example 2: Indeterminate Form of Type $\frac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{\ln x-1}{x-1}$$

Just to be clear, this isn't an indeterminate form because $x\to\infty$. It's because

$$\lim_{x\to\infty}\ln x-1=\infty \text{ and } \lim_{x\to\infty}x-1=\infty$$

Definition 1: L'Hosptial's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at $x = a$). Suppose that

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty$$

which gives us our two indeterminate forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

So what does this say?

It says the the limit of a quotient is equal to the limit of a quotient of their respected derivatives.

Warning! This is not the quotient rule! You are taking the derivatives of $f(x)$ and $g(x)$ separately.

Example 3

Evaluate

1. $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$
2. $\displaystyle\lim_{x\to \infty} \frac{\ln x - 1}{x - 1}$
3. $\displaystyle\lim_{t\to 0} \frac{e^{3t} - 1}{t^2}$

1. $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

Since this has the indeterminate form of type $\frac{0}{0}$, we can use L'Hospital's Rule.

$$\begin{aligned} \lim_{x\to 0}\frac{\sin(x)}{x} &\stackrel{LH}{=} \lim_{x\to 0}\frac{\cos(x)}{1} \\ &= \frac{\cos(0)}{1} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

2. $\lim_{x\to\infty}\frac{\ln x - 1}{x-1}$

This has the indterminate form of type $\frac{\infty}{\infty}$, so we can use L'Hospital's Rule.

$$\begin{aligned} \lim_{x\to\infty}\frac{\ln x - 1}{x-1} & stackrelLH= & \lim_{x\to\infty}\frac{\frac{1}{x}}{1} \\ &= & \lim_{x\to\infty}\frac{1}{x} \\ &= & 0 \end{aligned}$$

3. $\lim_{t\to 0}\frac{e^{3t}-1}{t^2}$

This has the indeterminate form of type $\frac{0}{0}$. You can use the rule as many times as you need as long as it has the appropriate indeterminate form.

$$\begin{aligned} \lim_{t\to 0}\frac{e^{3t}-1}{t^2} &\stackrel{LH}{=} \lim_{t\to 0}\frac{3e^{3t}}{2t} \\ &= \lim_{t\to 0}\frac{9e^{3t}}{2}^{***} \\ &= \frac{9e^{3\cdot 0}}{2} \\ &= 9/2 \end{aligned}$$

*** Did you notice I used L'Hospitals Rule a second time? Was I allowed to?

The answer is no. If you plug in $t = 0$, you get $\frac{3e^0}{0} = \frac{3}{0}$. This is not one of our indeterminate types. In this case, it means we can't use L'Hospitals Rule. We need to look at the left and right hand limits.

$$\lim_{t\to 0^-} \frac{3e^{3t}}{2t} = -\infty$$
$$\lim_{t\to 0^+} \frac{3e^{3t}}{2t} = \infty$$

Now we know how the function behaves at the asymptote.

So what did we learn here? Using L'Hospitals Rule is great (when you can use it). If you don't get one of the indeterminate forms, you can't apply the rule.

Definition 2: Indeterminate Products

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$. The limit $\lim_{x\to a} f(x)g(x)$ is called an indeterminate product.

We also call it **Indeterminate Form of type** $0 \cdot \infty$.

We have two ways of dealing with this type.

Steps 1: Rewriting an Indeterminate Product

1. Rewrite $f(x) \cdot g(x)$ as $\dfrac{f(x)}{1/g(x)}$.

   With this new form, we have an indeterminate form of type $\dfrac{0}{0}$. We now use L'Hospital's Rule.

2. Rewrite $f(x) \cdot g(x)$ as $\dfrac{g(x)}{1/f(x)}$

   With this form, we have an indeterminate form of type $\dfrac{\infty}{\infty}$. We now use L'Hopsital's Rule.

Example 4

Evaluate

1. $\displaystyle\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$ $\displaystyle\lim_{x\to-\infty} xe^x$

1. $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$

   Note that $x \to \infty$ and $\tan(1/x) \to 0$ as $x \to \infty$.

   We either write this as $\dfrac{\tan(1/x)}{1/x}$ or $\dfrac{x}{(\tan(1/x))^{-1}}$. The first option looks promising.

$$\begin{aligned}
\lim_{x\to\infty} x\tan(1/x) &= \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot -\frac{1}{x^2}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \sec^2(1/x) \\
&= \sec^2(0) \\
&= 1
\end{aligned}$$

2. $\lim_{x\to-\infty} xe^x$

   This has the indeterminate form of type $-\infty \cdot 0$

   We can either write xe^x as $\dfrac{e^x}{1/x}$ or $\dfrac{x}{e^{-x}}$

   I'd like you to try to use the first one. You'll find that the denominator won't go away.

   We are going to choose the second option.

$$\begin{aligned}
\lim_{x\to-\infty} xe^x &= \lim_{x\to-\infty} \frac{x}{e^{-x}} \\
&\overset{LH}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} \\
&= \lim_{x\to-\infty} -e^x \\
&= 0
\end{aligned}$$

Definition 3: Indeterminate Differences

Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then the limit

$$\lim_{x\to a} f(x) - g(x)$$

is called an **Indeterminate Form of Type** $\infty - \infty$. The doesn't mean the limit will be 0 or ∞. It is also possible that the limit is some finite number.

Example 5

Evaluate

1. $\lim\limits_{x\to\infty} \sqrt{x^2+x} - x$
2. $\lim\limits_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

1. $\lim\limits_{x\to\infty} \sqrt{x^2+x} - x$

   This has the indeterminate form of type $\infty-\infty$. Our approach is to simplify somehow. We can use common denominators, conjugates, factoring, pretty much anything that gets us to one single fraction.

   In this case, we'll multiply top and bottom by the conjugate.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+x} - x &= \lim_{x\to\infty} \sqrt{x^2+x} - x \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{(x^2+x) - x^2}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} \\
&= \lim_{x\to\infty} \frac{x}{\frac{1}{x}\left(\sqrt{1+\frac{1}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}+1}} \\
&= \frac{1}{2}
\end{aligned}$$

2. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

This has the indeterminate form of $\infty - \infty$.

Again, we would like to make this into one fraction. That way, we can hope for $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) &= \lim_{x\to 0^+}\frac{e^x-1}{x(e^x-1)}-\frac{x}{x(e^x-1)} \\
&= \lim_{x\to 0^+}\frac{e^x-x-1}{x(e^x-1)} \\
&\overset{LH}{=} \lim_{x\to 0^+}\frac{e^x-1}{xe^x+(e^x-1)} \\
&\overset{LH}{=} \lim_{x\to 0^+}\frac{e^x}{xe^x+e^x+e^x} \\
&= \lim_{x\to 0^+}\frac{e^x}{xe^x+2e^x} \\
&\overset{LH}{=} \lim_{x\to 0^+}\frac{1}{x+2} \\
&= \frac{1}{2}
\end{aligned}$$

Definition 4: Indeterminate Powers

Consider the following limit

$$\lim_{x\to a}(f(x))^{g(x)}$$

1. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then we have an indeterminate form of type 0^0.
2. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$, then we have an indeterminate form of type ∞^0.
3. If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$, then we have an indeterminate form of type 1^∞.

Here's how we approach these types. Instead of working with

$$y = (f(x))^{g(x)}$$

we work on

$$\ln y = g(x) \cdot \ln(f(x))$$

From here it's likely we have one of our previous indeterminate forms.

Example 6

Evaluate

1. $\lim\limits_{x\to\infty} x^{1/x}$
2. $\lim\limits_{x\to 0^+} (1+\sin(4x))^{\cot(x)}$

1. $\lim\limits_{x\to\infty} x^{1/x}$

   We have the indeterminate form of type ∞^0

$$\text{Let } \ln y = \ln\left(x^{1/x}\right) = \frac{1}{x}\cdot\ln(x)$$

   Now let's find the limit

$$\lim_{x\to\infty} \frac{1}{x}\ln(x)$$

$$\begin{aligned}\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{1}{x}\ln(x)\\ &= \lim_{x\to\infty} \frac{\ln(x)}{x}\\ &= \lim_{x\to\infty} \frac{1/x}{1}\\ &= 0\end{aligned}$$

   So as $x \to \infty$, we have $\ln y \to 0$. If we exponentiate both sides

$$e^{\ln y} \to e^0$$

$$y \to 1$$

Therefore, $\lim\limits_{x\to\infty} x^{1/x} = 1$

2. $\lim\limits_{x\to 0^+} (1+\sin(4x))^{\cot(x)}$

   Now we have type 1^∞

   Instead of $\lim y = \lim(1+\sin(4x))^{\cot(x)}$, let's work on

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot(x)\ln(1+\sin(4x))$$

$$\begin{aligned}
\lim_{x\to 0^+} \cot(x)\ln(1+\sin(4x)) &= \lim_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\tan(x)} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(4x)} \cdot 4\cos(4x)}{\sec^2(x)} \\
&= \frac{\frac{1}{1+\sin(0)} \cdot 4\cos(0)}{\sec^2(0)} \\
&= 4
\end{aligned}$$

   If $\ln y \to 4$ as $x \to 0^+$, then $y \to e^4$ as $x \to 0^+$.

   Therefore,

$$\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)} = e^4$$