Math 230

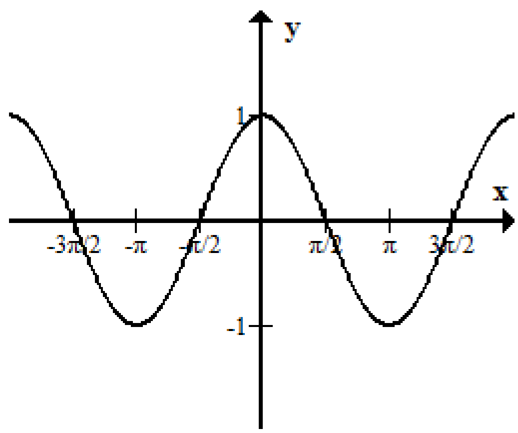
Calculus II

Brian Veitch • Fall 2015 • Northern Illinois University

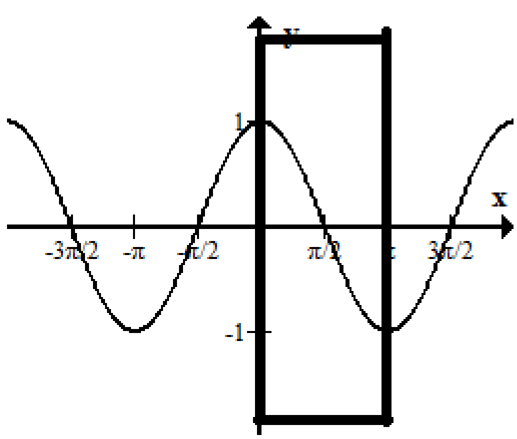
Inverse Trig Functions

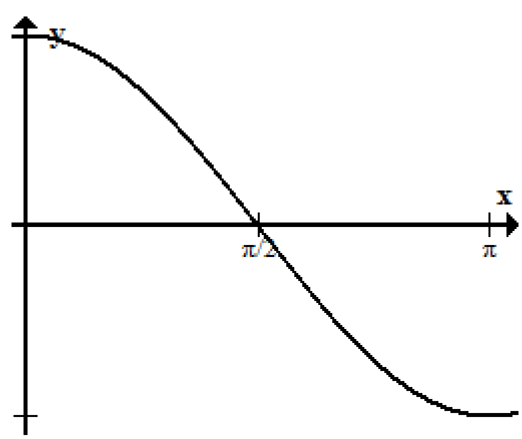
Trig functions are not one to one, so they do not have inverse functions. To deal with this, just like how we dealt with $f(x) = x^2$, we must restrict the trig function's domain.

Let's take a look at $f(x) = \cos(x)$



$f(x) = \cos(x)$ does not pass the horizontal line test. But if we restrict the domain, it's possible to make it one to one.





If we restrict the domain to $[0, \pi]$, then it passes the horizontal line test, and therefore has an inverse.

By definition, the inverse is

$$\cos^{-1} x = y \quad \Rightarrow \quad \cos(y) = x, \text{ where } 0 \leq y \leq \pi$$

Example 1

Find $\cos^{-1}(\sqrt{2}/2)$

1. To solve this we need to find what value of x gives us

$$\cos(x) = \sqrt{2}/2$$

2. Since $\cos(\pi/4) = \sqrt{2}/2$, our answer is

$$\cos^{-1}(\sqrt{2}/2) = \pi/4$$

Example 2

Find $\cos^{-1}(\cos(-\pi))$

1. Even though these are inverse functions, they just don't cancel leaving us with $-\pi$. What I mean,

$$\cos^{-1}(\cos(-\pi)) \neq -\pi$$

By definition,

$$\cos^{-1}(\cos(-\pi)) = Y$$

is the same as

$$\cos(Y) = \cos(-\pi), \text{ where } 0 \leq Y \leq \pi$$

Since $\cos(-\pi) = -1$, where in $0 \leq Y \leq \pi$ does $\cos(Y) = -1$. That is at $Y = \pi$.

Now on to the other inverse trig functions.

$$\sin^{-1}(x) = y \quad \Rightarrow \quad \sin(y) = x \text{ where } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\tan^{-1}(x) = y \quad \Rightarrow \quad \tan(y) = x \text{ where } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Note the inequalities on the range of $\tan^{-1}$, i.e, y. They are strictly less than signs since $\tan(\pi/2)$ and $\tan(-\pi/2)$ do not exist.

Example 3

Find $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Set it up like this,

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = y$$

so,

$$\tan(y) = \frac{1}{\sqrt{3}}$$

Where does this occur on the unit circle between $-\frac{\pi}{2}$ to $\frac{\pi}{2}$?

$$\tan(\pi/6) = \frac{1}{\sqrt{3}}$$

therefore,

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

Example 4

Find $\csc^{-1}(2)$

Set it up like this,

$$\csc^{-1}(2) = y$$

where,

$$\csc(y) = 2$$

But using $\csc(y)$ is hard. Let's change that to $\dfrac{1}{\sin(x)}$

$$\frac{1}{\sin(y)} = 2 \quad \to \sin(y) = \frac{1}{2}$$

So where in the interval $[-\pi/2, \pi/2]$, does $\sin(y) = \dfrac{1}{2}$

$$\sin(\pi/6) = \frac{1}{2}$$

therefore,

$$\csc^{-1}(2) = \pi/6$$

Formula 1: Derivative of the Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}} \text{ where } -1 < x < 1$$

$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}} \text{ where } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1}(x)) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$$

Example 5

1. $\frac{d}{dx}\left[\sqrt{\tan^{-1}(x)}\right]$
2. $\frac{d}{dx}\left[\sqrt{x^2-1}\cdot\sec^{-1}(3x)\right]$
3. $\frac{d}{dx}\left[\frac{1}{\cos^{-1}(x)}\right]$

1. $\frac{d}{dx}\left[\sqrt{\tan^{-1}(x)}\right]$

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{\tan^{-1}(x)}\right] &= \frac{1}{2}(\tan^{-1}(x))^{-1/2}\cdot\frac{d}{dx}\left[\tan^{-1}(x)\right] \\ &= \frac{1}{2}(\tan^{-1}(x))^{-1/2}\cdot\frac{1}{1+x^2}\end{aligned}$$

2. $\frac{d}{dx}\left[\sqrt{x^2-1}\cdot\sec^{-1}(3x)\right]$

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{x^2-1}\cdot\sec^{-1}(3x)\right] &= \sqrt{x^2-1}\cdot\frac{d}{dx}\left[\sec^{-1}(3x)\right]+\sec^{-1}(3x)\cdot\frac{d}{dx}\left[\sqrt{x^2-1}\right]\\ &= \sqrt{x^2-1}\cdot\frac{1}{(3x)\sqrt{(3x)^2-1}}\cdot(3)+\sec^{-1}(3x)\cdot\frac{1}{2}(x^2-1)^{-1/2}\cdot(2x)\\ &= \frac{\sqrt{x^2-1}}{3x\sqrt{(3x)^2-1}}+\frac{x\sec^{-1}(3x)}{\sqrt{x^2-1}}\end{aligned}$$

3. $\frac{d}{dx}\left[\frac{1}{\cos^{-1}(x)}\right]$

Note: $\frac{1}{\cos^{-1}(x)} = (\cos^{-1}(x))^{-1}$

Be careful that you don't confuse the -1s. One is the notation for trig inverse, and the other is an exponent of -1.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{\cos^{-1}(x)}\right] &= \frac{d}{dx}\left[\left(\cos^{-1}(x)\right)^{-1}\right]\\ &= -1\left(\cos^{-1}(x)\right)^{-2}\cdot\frac{d}{dx}\left[\cos^{-1}(x)\right]\\ &= \frac{1}{(\cos^{-1}(x))^2\sqrt{1-x^2}}\end{aligned}$$

Integrals of Inverse Trig Functions

Example 6

1. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$
2. $\int \frac{1}{x\sqrt{x^2-4}}\,dx$
3. $\int \frac{x}{x^4+9}$

1. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$

This is a pretty straightforward substitution

(a) Let $u = \tan^{-1}(x)$

(b) $du = \dfrac{1}{1+x^2}\, dx$

(c) Now substitute

$$\begin{aligned}\int \frac{\tan^{-1} x}{1+x^2}\, dx &= \int u\, du \\ &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}\left(\tan^{-1} x\right)^2 + C\end{aligned}$$

2. $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\, dx$

We know that $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1}(x)$, but that's not quite what we have. Do you see the 4 in $\sqrt{x^2-4}$? It needs to be a 1. So here's what we do

(a) Factor out 4

$$\frac{1}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}}$$

$$\frac{1}{2x\sqrt{\left(\frac{x}{2}\right)^2-1}}$$

See? Now we have the correct form of $u^2 - 1$.

(b) Let $u = \dfrac{x}{2}$. Note that $x = 2u$ for the denominator.

(c) $du = \dfrac{1}{2}\, dx$

(d) Now substitute

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-4}}\, dx &= \int \frac{1}{2u\sqrt{u^2-1}}\, du \\ &= \frac{1}{2}\sec^{-1}(u) + C \\ &= \frac{1}{2}\sec^{-1}(x/2) + C\end{aligned}$$

3. $\int \frac{x}{x^4+9}$

(a) If the denominator looked like x^2+9, it would have the similar form for $\tan^{-1}$.

(b) Let $u = x^2$

(c) $du = 2x\ dx \rightarrow \frac{1}{2}\ du = x\ dx$

(d) Substitute

$$\int \frac{x}{x^4+9} = \frac{1}{2}\int \frac{1}{u^2+9}\ du$$

WAIT!! What about the 9 in u^2+9. It's suppose to be u^2+1. Ugh.

(e) Factor out 9

$$\frac{1}{9\left(\frac{u^2}{9}+1\right)}$$

$$\frac{1}{9\left(\left(\frac{u}{3}\right)^2+1\right)}$$

(f) Oh boy, another substitution

(g) Let $w = \frac{u}{3}$

(h) $dw = \frac{1}{3}\ du \rightarrow 3\ dw = du$

$$\begin{aligned}
\frac{1}{2}\int \frac{1}{9\left(\left(\frac{u}{3}\right)^2+1\right)}\ du &= \frac{1}{2}\frac{1}{9}\int \frac{1}{w^2+1}\ (3\ dw) \\
&= \frac{1}{6}\int \frac{1}{w^2+1}\ dw \\
&= \frac{1}{6}\tan^{-1}(w) \\
&= \frac{1}{6}\tan^{-1}\left(\frac{u}{3}\right) \\
&= \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C
\end{aligned}$$