# Math 230

## Calculus II

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## General Logarithmic and Exponential Functions

Up to this point, we've only dealt with the exponential function $e^x$ and the log function $\ln x$. These two functions have base $e$. So what about all those exponential and log functions with a different base? What's their derivative and integral?

**Rule 1**

$$a^x = e^{x \ln a}$$

If $x$ and $y$ are real numbers and $a, b > 0$, then

1. $a^{x+y} = a^x a^y$
2. $a^{x-y} = \dfrac{a^x}{a^y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

**Definition 1: Derivative for $a^x$**

$$\frac{d}{dx}(a^x) = a^x \ln a$$

We will use the definition from the beginning of this section.

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{x\ln a}\\ &= e^{x\ln a}\frac{d}{dx}(x\ln a)\\ &= e^{x\ln a}\ln a\\ &= a^x\ln a\end{aligned}$$

The biggest problem students face with this section is where does $\ln a$ go. In fact, I sometimes find it easier to change $a^x$ to $e^{x\ln a}$.

**Example 1**

Differentiate the following

1. $y = 5^x(1+5\ln x)$
2. $y = 2x\cdot 5^{1/x}$
3. Use logarithmic differentiation to find $y'$ when $y = x^{\sqrt{x}}$

1. $y = 5^x(1+5\ln x)$

$$\begin{aligned}y' &= 5^x\cdot\frac{d}{dx}(1+5\ln x) + (1+5\ln x)\cdot\frac{d}{dx}5^x\\ &= 5^x(1+\frac{5}{x}) + (1+5\ln x)\cdot 5^x\cdot\ln 5\end{aligned}$$

2. $y = 2x\cdot 5^{1/x}$

$$\begin{aligned}y' &= 2x\cdot\frac{d}{dx}(5^{1/x}) + 5^{1/x}\cdot\frac{d}{dx}(2x)\\ &= 2x\cdot\left(5^{1/x}\ln 5\cdot -\frac{1}{x^2}\right) + 5^{1/x}\cdot 2\\ &= \frac{2\ln 5\cdot 5^{1/x}}{x} + 2\cdot 5^{1/x}\end{aligned}$$

3. Use logarithmic differentiation to find $y'$ when $y = x^{\sqrt{x}}$

(a) Take ln of both sides.

$$\ln y = \ln x^{\sqrt{x}}$$

$$\ln y = \sqrt{x} \ln x$$

(b) Differentiate both sides with respect to $x$

$$\begin{aligned} \frac{1}{y} \cdot y' &= \sqrt{x} \cdot \frac{1}{x} + \ln(x) \cdot \frac{1}{2\sqrt{x}} \\ y' &= y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) \\ y' &= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \end{aligned}$$

(c) OR rewrite function as $y = e^{\sqrt{x}\ln x}$ and do the chain rule

$$\begin{aligned} y' &= e^{\sqrt{x}\ln x} \cdot \frac{d}{dx}\left(\sqrt{x}\ln x\right) \\ &= e^{\sqrt{x}\ln x} \cdot \left(\frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x}\right) \\ &= e^{\sqrt{x}\ln x} \cdot \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right) \\ y' &= x^{\sqrt{x}}\left(\frac{2 + \ln x}{2\sqrt{x}}\right) \end{aligned}$$

**Definition 2: Evaluating $\int a^x \, dx$**

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

**Example 2**

Find $\int_0^2 5^x \, dx$

$$\begin{aligned}\int_0^2 5^x\,dx &= \left.\frac{5^x}{\ln 5}\right|_0^2\\ &= \frac{5^2}{\ln 5}-\frac{5^0}{\ln 5}\\ &= \frac{24}{\ln 5}\end{aligned}$$

**Example 3**

Find $\int 8^{\tan x}\cdot \sec^2(x)\,dx$

1. Let $u = \tan(x)$
2. $du = \sec^2(x)\,dx$
3. Substitute

$$\begin{aligned}\int 8^{\tan(x)}\cdot\sec^2(x)\,dx &= \int 8^u\,du\\ &= \frac{8^u}{\ln 8}+C\\ &= \frac{8^{\tan(x)}}{\ln 8}+C\end{aligned}$$

**Definition 3: Derivative of $\log_a x$**

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$
$$\frac{d}{dx}(\log_a u) = \frac{1}{u\ln a}\cdot\frac{du}{dx}$$

So if you were asked to find $\log_8 5$, you evaluate it as $\frac{\ln 5}{\ln 8} = 0.773976$ on your calculator.

**Example 4**

Differentiate the following

1. $y = \log_5(xe^x)$

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   There are two ways of doing this problem. Recall that when you're taking the derivative of a log, check to see if you can break it up into smaller logs.

   (a) Straightforward Derivative

$$\begin{aligned} y' &= \frac{1}{xe^x \ln 5} \cdot \frac{d}{dx}(xe^x) \\ &= \frac{1}{xe^x \ln 5} \cdot (xe^x + e^x) \\ &= \frac{1}{xe^x \ln 5} \cdot e^x(x+1) \\ &= \frac{e^x(x+1)}{xe^x \ln 5} \\ &= \frac{x+1}{x \ln 5} \end{aligned}$$

   (b) Or, rewrite $\log_5(xe^x)$ as $y = \dfrac{\ln(xe^x)}{\ln 5} = \dfrac{\ln x + \ln e^x}{\ln 5} = \dfrac{1}{\ln 5}(\ln x + x)$

$$y' = \frac{1}{\ln 5}\left(\frac{1}{x} + 1\right)$$
$$y' = \frac{1}{\ln 5} \cdot \frac{x+1}{x}$$

   Wasn't that so much easier??

**Example 5**

Let's do some more integrals.

1. $\displaystyle\int \frac{\log_{10}(x+1)}{x+1}\,dx$
2. $\displaystyle\int \frac{3^x}{3^x+1}\,dx$

1. $\displaystyle\int \frac{\log_{10}(x+1)}{x+1}\,dx$
   (a) Let $u = \log_{10}(x+1)$
   (b) $du = \dfrac{1}{(x+1)\ln 10}\,dx \to \ln(10)\,du = \dfrac{dx}{x+1}$
   (c) Substitute

$$\begin{aligned}\int \frac{\log_{10} x+1}{x+1}\,dx &= \int u\cdot\ln(10)\,du\\ &= \frac{1}{2}u^2\ln(10)+C\\ &= \frac{\ln 10}{2}\cdot(\log_{10}(x+1))^2+C\end{aligned}$$

2. $\displaystyle\int \frac{3^x}{3^x+1}\,dx$
   (a) Let $u = 3^x+1$
   (b) $du = 3^x\cdot\ln 3\,dx \to \dfrac{1}{\ln 3}\,du = 3^x\,dx$
   (c) Substitute

$$\begin{aligned}\int \frac{3^x}{3^x+1}\,dx &= \int \frac{1}{u}\cdot\frac{1}{\ln 3}\,du\\ &= \frac{1}{\ln 3}\cdot\ln u+C\\ &= \frac{1}{\ln 3}\cdot\ln(3^x+1)+C\end{aligned}$$