Math 230

Calculus II

Brian Veitch • Fall 2015 • Northern Illinois University

The Natural Exponential Function

How do the exponential function and the ln function relate to each other and what are some of their properties?

1. If $y = e^x$, then $\ln y = x$
2. $e^{\ln x} = x$ and $\ln e^x = x$

This shows that $\ln x$ and e^x are inverse functions.

Example 1

Find x if $\ln x = 5$.

Using (1), if $\ln x = 5$, then $e^5 = x$. And that's our solution. $x = e^5$. That was easy!

Example 2

Solve the equation $e^{5-3x} = 10$

1. Start by taking ln of both sides

$$e^{5-3x} = 10$$
$$\ln\left(e^{5-3x}\right) = \ln(10)$$

2. Property (2b) gives us

$$5 - 3x = \ln(10)$$

3. Now we simply solve for x

$$\begin{aligned} 5-3x &= \ln(10) \\ -3x &= \ln(10)-5 \\ x &= \frac{1}{3}(5-\ln(10)) \end{aligned}$$

Example 3

Let $f(x)=\sqrt{3-e^{2x}}$. Find f^{-1}, and the domain and range of f.

1. To be in the domain we need $3-e^{-2x}\geq 0$.

$$\begin{aligned} 3-e^{2x} &\geq 0 \\ -e^{2x} &\geq -3 \\ e^{2x} &\leq 3 \\ 2x &\leq \ln 3 \\ x &\leq \frac{\ln 3}{2} \end{aligned}$$

2. To find the range it's easier to find the domain of f^{-1}.

   (a) Let $y=\sqrt{3-e^{2x}}$

   (b) Switch x and y to get $x=\sqrt{3-e^{2y}}$

   (c) Solve for y

$$\begin{aligned} x &= \sqrt{3-e^{2y}} \\ x^2 &= 3-e^{2y} \\ x^2-3 &= -e^{2y} \\ 3-x^2 &= e^{2y} \\ \ln(3-x^2) &= 2y \\ \frac{\ln(3-x^2)}{2} &= y \end{aligned}$$

$$f^{-1}(x) = \frac{\ln(3 - x^2)}{2}$$

(d) To find the domain of f^{-1} we need to solve $3 - x^2 > 0$ which occurs when $-\sqrt{3} < x < \sqrt{3}$.

(e) The final answer for the range, however, is $[0, \sqrt{3}]$. This happens because the original function $f(x)$ has a $\sqrt{\ }$ so it can't take on any negative values.

Definition 1: Properites of the Natural Exponential Function

1. The exponential function $f(x) = e^x$ is an increasing continuous function with domain R and range $(0, \infty)$.

2. $\lim_{x \to -\infty} e^x = 0$

3. $\lim_{x \to \infty} e^x = \infty$

Example 4

Using the above properties, find

$$\lim_{x \to 2^-} e^{\frac{3}{2-x}}$$

1. Let's see if we can rewrite this so it looks like one of our properites.

   Let $t = \dfrac{3}{2 - x}$

2. As $x \to 2^-$, $t \to \infty$.

3. Let's rewrite the original limit

$$\lim_{x \to 2^-} e^{\frac{3}{2-x}}$$

$$\lim_{t \to \infty} e^t = \infty$$

Definition 2: Derviative of $f(x) = e^x$

$$\frac{d}{dx}(e^x) = e^x$$

The proof is pretty simple. Since we know the from one of our earlier properties that if $y = e^x$, then $\ln y = x$. Now, let's differentiate this with respect to x. Keep in mind, this is now an implicit function.

$$\begin{aligned} \frac{d}{dx}\ln y &= \frac{d}{dx}x \\ \frac{1}{y}\cdot y' &= 1 \\ y' &= y \\ y' &= e^x \end{aligned}$$

Also, using the chain rule, we get

$$\frac{d}{dx}e^u = e^u \cdot \frac{du}{dx}$$

Let's do some derivative problems, shall we?

Example 5

Differentiate

1. $f(x) = (x^3 + 2x)e^{x^2}$
2. $f(x) = \sin(e^x) + e^{\sin(x)}$
3. $f(x) = \sqrt{1 + xe^{-2x}}$
4. $f(x) = \cos\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right)$

1. $f(x) = (x^3 + 2x)e^{x^2}$

$$\begin{aligned} f'(x) &= (x^3 + 2x) \cdot \frac{d}{dx}(e^{x^2}) + e^{x^2} \cdot \frac{d}{dx}(x^3 + 2x) \\ &= (x^2 + 2x) \cdot e^{x^2} \cdot (2x) + e^{x^2} \cdot (3x^2 + 2) \\ &= e^{x^2}(x^3 + 2x + 3x^2 + 2) \end{aligned}$$

2. $f(x) = \sin(e^x) + e^{\sin(x)}$

$$\begin{aligned} f'(x) &= \cos(e^x) \cdot \frac{d}{dx}(e^x) + e^{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) \\ &= \cos(e^x) \cdot e^x + e^{\sin(x)} \cdot \cos(x) \end{aligned}$$

3. $f(x) = \sqrt{1 + xe^{-2x}}$

Rewrite $f(x)$ as $f(x) = (1 + xe^{-2})^{1/2}$

$$\begin{aligned} f'(x) &= \frac{1}{2}(1 + xe^{-2x})^{-1/2} \cdot \frac{d}{dx}(1 + xe^{-2x}) \\ &= \frac{1}{2}(1 + xe^{-2x})^{-1/2} \cdot (xe^{-2x} \cdot (-2) + e^{-2x} \cdot (1)) \\ &= \frac{1}{2}(1 + xe^{-2x})^{-1/2} \cdot (-2xe^{-2x} + e^{-2x}) \\ &= \frac{e^{-2x}(-2x + 1)}{2(1 + xe^{-2x})} \end{aligned}$$

4. $f(x) = \cos\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right)$

$$\begin{aligned} f'(x) &= -\sin\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \cdot \frac{d}{dx}\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \\ &= -\sin\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \cdot \left(\frac{(1 + e^{2x}) \cdot (-2e^{2x}) - (1 - e^{2x}) \cdot (2e^{2x})}{(1 + e^{2x})^2}\right) \\ &= -\sin\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \cdot \left(\frac{-2e^{2x} - 2e^{4x} - 2e^{2x} + 2e^{4x}}{(1 + e^{2x})^2}\right) \\ &= -\sin\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \cdot \left(\frac{-4e^{2x}}{(1 + e^{2x})^2}\right) \\ &= \sin\left(\frac{1 - e^{2x}}{1 + e^{2x}}\right) \cdot \left(\frac{4e^{2x}}{(1 + e^{2x})^2}\right) \end{aligned}$$

Let's go ahead and try some integrals. Here's our integration formula.

Definition 3: Evaluating $\int e^u\,du$

$$\int e^u\,du = e^u + C$$

Example 6

Integrate the following

1. $\displaystyle\int_0^1 xe^{-x^2}\,dx$
2. $\displaystyle\int e^x\sqrt{1+e^x}$
3. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$

1. $\displaystyle\int_0^1 xe^{-x^2}\,dx$
   (a) Let $u = -x^2$
   (b) $du = -2x\,dx \to \dfrac{-du}{2} = x\,dx$
   (c) Change the bounds

$$\text{If } x = 0, u = 0$$

$$\text{If } x = 1, u = -1$$

   (d) Substitute

$$\begin{aligned}\int_0^1 xe^{-x^2}\,dx &= \int_0^{-1} -\frac{1}{2}e^u\,du \\ &= -\frac{1}{2}e^u\Big|_0^{-1} \\ &= -\frac{1}{2}e^{-1} + \frac{1}{2}e^0 \\ &= \frac{1}{2}(1 - e^{-1})\end{aligned}$$

2. $\int e^x\sqrt{1+e^x}$

   (a) Let $u = 1 + e^x$

   (b) $du = e^x\ dx$

   (c) Substitute

$$\begin{aligned}\int e^x\sqrt{1+e^x}\ dx &= \int \sqrt{u}\ du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(1+e^x)^{3/2} + C\end{aligned}$$

3. $\int \frac{e^{1/x}}{x^2}\ dx$

   (a) Let $u = 1/x$

   (b) $du = -\frac{1}{x^2}\ dx \to -du = \frac{dx}{x^2}$

   (c) Substitute

$$\begin{aligned}\int \frac{e^{1/x}}{x^2}\ dx &= \int e^u(-du) \\ &= -e^u + C \\ &= -e^{1/x} + C\end{aligned}$$