Math 230

Calculus II

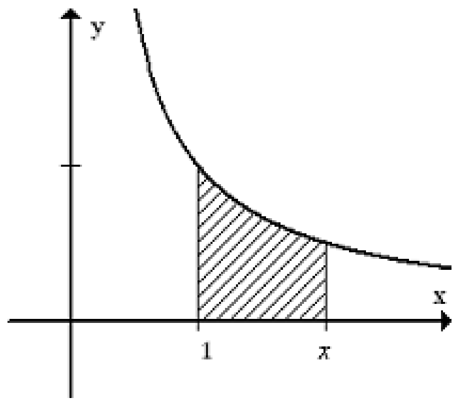
Brian Veitch • Fall 2015 • Northern Illinois University

Natural Logarithmic Function

Definition 1: The Natural Logarithm

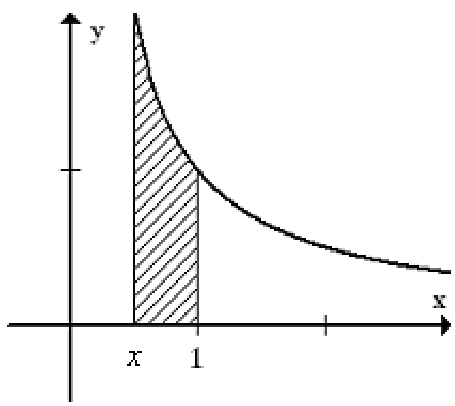
The natural logarithmic function is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt$$



So $\ln x$ is the area under the curve $f(x) = \dfrac{1}{t}$ from 1 to x.

For $0 < x < 1$:



This means the value of $\ln x$, $x < 1$, is the negative of the area shown above. This should make sense from one of our integration properties.

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

Definition 2: Derivative of the Natural Logarithm

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

It comes directly from the definition of $\ln x$ and the Fundamental Theorem of Calc Part 1.

$$\frac{d}{dx}\ln x = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

Laws of Logarithms

Rules 1: Laws of Logarithms

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
3. $\ln(\left(\frac{1}{x}\right) = -\ln(x)$
4. $\ln(x^r) = r\ln x$

Example 1

Expand $\ln \frac{3x^2}{(x+1)^3}$

$$\begin{aligned}\ln \frac{3x^2}{(x+1)^3} &= \ln 3x^2 - \ln(x+1)^3 \\ &= \ln 3x^2 - 3\ln(x+1)\end{aligned}$$

Example 2

Expand $\ln \dfrac{(x^2+5)^{10}\sqrt{\cos(x)}}{x^3}$

$$\begin{aligned}\ln \frac{(x^2+5)^{10}\sqrt{\cos(x)}}{x^3} &= \ln(x^2+5)^{10} + \ln\sqrt{\cos(x)} - \ln x^3 \\ &= 10\ln(x^2+5) + \frac{1}{2}\ln\cos(x) - 3\ln x\end{aligned}$$

Example 3

Express as one ln: $\dfrac{1}{3}\ln(1+x^2) - \ln x - 2\ln\sin(x)$

$$\frac{1}{3}\ln(1+x^2) - \ln x - 2\ln\sin(x) = \ln\left(\frac{(1+x^2)^{1/3}}{x\sin^2(x)}\right)$$

Definition 3: Derivative of the Natural Logarithm

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{1}{g(x)}\cdot g'(x)$$

Example 4

Find $\dfrac{d}{dx}\ln(\tan x)$:

$$\begin{aligned}\frac{d}{dx}\ln(\tan x) &= \frac{1}{\tan(x)}\cdot\frac{d}{dx}\left[\tan(x)\right] \\ &= \frac{1}{\tan(x)}\cdot\sec^2(x)\end{aligned}$$

Example 5

Find $\frac{d}{dx}\ln\left(\frac{x+1}{\sqrt{x-2}}\right)$

You can do this in two ways. We'll do both though. The first way, we just differentiate using our current log derivative rule.

1. Take the derivative directly using the chain rule.

$$\begin{aligned}
\frac{d}{dx}\ln\left(\frac{x+1}{\sqrt{x-2}}\right) &= \frac{1}{\frac{x+1}{\sqrt{x-2}}}\cdot\frac{d}{dx}\left(\frac{x+1}{\sqrt{x-2}}\right) \\
&= \frac{\sqrt{x-2}}{x+1}\cdot\left(\frac{\sqrt{x-2}\cdot 1-(x+1)\cdot\frac{1}{2}(x-2)^{-1/2}}{(x-2)}\right) \\
&= \frac{(x-2)-\frac{1}{2}(x+1)}{(x+1)(x-2)} \\
&= \frac{\frac{1}{2}x-\frac{3}{2}}{(x+1)(x-2)}
\end{aligned}$$

2. Rewrite $\ln\left(\frac{x+1}{\sqrt{x-2}}\right)$ as $\ln(x+1)-\frac{1}{2}\ln(x-2)$

   Now differentiate,

$$\frac{d}{dx}\ln\left(\frac{x+1}{\sqrt{x-2}}\right)=\frac{1}{x+1}-\frac{1}{2(x-2)}$$

Which one do you think was easier?

Example 6

Find $\frac{d}{dx}\left(\ln(\tan(x))\right)^2$

$$y' = 2\left(\ln\tan(x)\right)\cdot\frac{1}{\tan(x)}\cdot\sec^2(x)$$

Example 7

Find $\dfrac{d}{dx}\left[x^2 \ln\left(\dfrac{1}{x^2}\right)\right]$

Use the product rule

$$\begin{aligned}
\frac{d}{dx}\left[x^2 \ln\left(\frac{1}{x^2}\right)\right] &= x^2 \cdot \left(x^2 \cdot -\frac{2}{x^3}\right) + 2x \cdot \ln\left(\frac{1}{x^2}\right) \\
&= -2x + 2x \cdot \ln\left(\frac{1}{x^2}\right) \\
&= -2x(1 - 2\ln(x))
\end{aligned}$$

How could you make this problem a bit easier? You could rewrite the original problem.

Note, $x^2 \ln\left(\dfrac{1}{x^2}\right) = -2x^2 \ln(x)$, which is far easier to differentiate.

$$-2x^2 \cdot \frac{1}{x} + (-4x)\ln(x) = -2x - 4x\ln(x) = -2x(1 - 2\ln(x))$$

Example 8

Given $f(x) = \ln|x|$, show $f'(x) = \dfrac{1}{x}$.

$$\ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

After you differentiate the piece-wise function, you get

$$f'(x) = \begin{cases} \dfrac{1}{x}, & x > 0 \\ \dfrac{1}{-x} \cdot (-1) = \dfrac{1}{x}, & x < 0 \end{cases}$$

So what does this mean?

1. $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$
2. $\displaystyle\int \frac{1}{x}\, dx = \ln |x| + C$

Logarithmic Differentiation

We use this method when we need to take derivatives of complicated products or quotients.

Example 9: Logarithmic Differentiation

Differentiate $y = \dfrac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$

1. Take ln of both sides.

$$\begin{aligned} \ln y &= \ln\left(\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\right) \\ &= \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \end{aligned}$$

   This looks a bit nicer, right?

2. Differentiate both sides with respect to x

$$\begin{aligned} \frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x} + \frac{1}{2(x^2+1)}\cdot 2x - \frac{2}{3(x+1)} \\ \frac{1}{y}\cdot\frac{dy}{dx} &= \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)} \end{aligned}$$

   Beats doing the quotient rule with a product rule and a chain rule!

3. Solve for $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)\\ &= \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\cdot\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)\end{aligned}$$

I dare you to try this without logarithmic differentiation.

Let's move on to integration using $\ln x$.

Since we know $\displaystyle\int \frac{1}{x}\,dx = \ln|x|$, let's do some examples.

Example 10

Find $\displaystyle\int \frac{2x}{x^2-5}\,dx$

1. Let $u = x^2 - 5$
2. $du = 2x\ dx$
3. Make the substitution

$$\begin{aligned}\int \frac{2x}{x^2-5}\,dx &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln(x^2-5) + C\end{aligned}$$

Example 11

Find $\displaystyle\int \frac{4\cos(x)}{3+2\sin(x)}\,dx$

1. Let $u = 3 + 2\sin(x)$
2. $du = 2\cos(x)\ dx \rightarrow 2du = 4\cos(x)\ dx$

3. Make the substitution

$$\begin{aligned}
\int \frac{4\cos(x)}{3+2\sin(x)}\,dx &= \int \frac{2}{u}\,du \\
&= 2\ln|u| + C \\
&= 2\ln(3+2\sin(x)) + C
\end{aligned}$$

Example 12

Find $\int \tan(x)\,dx$

1. We can't do much with $\tan(x)$. Let's rewrite $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
2. So $\int \tan(x)\,dx = \int \dfrac{\sin(x)}{\cos(x)}\,dx$
3. Let $u = \cos(x)$
4. $du = -\sin(x)\,dx \to -du = \sin(x)\,dx$
5. Make the substitution

$$\begin{aligned}
\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx \\
&= \int \frac{1}{u}\,(-du) \\
&= \int -\frac{1}{u}\,du \\
&= -\ln|u| + C \\
&= -\ln|\cos(x)| + C \\
&= \ln|(\cos(x))^{-1}| + C \\
&= \ln|\sec(x)| + C
\end{aligned}$$

Example 13

Evaluate $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$

This problem doesn't have any trick to it. We just need to break it up into two fractions.

1. Break it up into two separate fractions
$$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \frac{4}{u^3} + \frac{1}{u}\,du$$
2. At this point, just integrate like you normally would.
$$\begin{aligned}\int_1^2 \frac{4}{u^3} + \frac{1}{u}\,du &= \int_1^2 4u^{-3} + \frac{1}{u}\,du\\ &= -2u^{-2} + \ln|u|\Big|_1^2\\ &= \left(\frac{-2}{2^2} + \ln(2)\right) - \left(\frac{-2}{1^1} + \ln(1)\right)\\ &= \frac{3}{2} + \ln(2)\end{aligned}$$

Example 14

Evaluate $\displaystyle\int_e^6 \frac{dx}{x\ln x}$

1. This is a substitution problem.
2. Let $u = \ln(x)$.
3. Why $u = \ln(x)$? Because $du = \dfrac{1}{x}\,dx$ and that's also in the integral.
4. Since it's a definite integral, we need to change the limits of integration
$$\text{If } x = e, u = \ln(e) = 1$$
$$\text{If } x = 6, u = \ln(6)$$

5. Make the substitution

$$\begin{aligned}
\int_e^6 \frac{dx}{x\ln x} &= \int_1^{\ln(6)} \frac{1}{u}\,du \\
&= \ln|u|\Big|_1^{\ln(6)} \\
&= \ln|\ln(6)| - \ln|\ln(1)| \\
&= \ln|\ln(6)|
\end{aligned}$$