Math 230

Calculus II

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Inverse Functions: Exponential, Log, and Inverse Trig

Consider the following functions,

1. $f(x) = 2x + 3$, has an inverse function

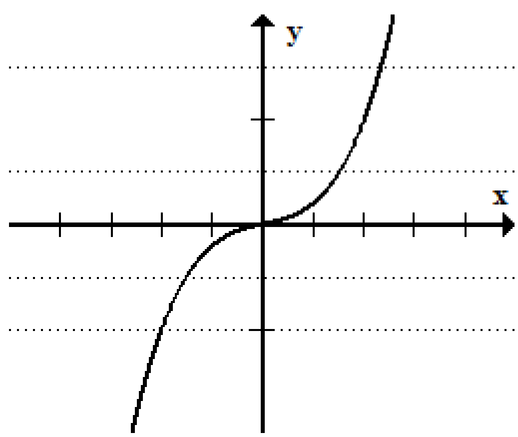
2. $f(x) = x^2$, has no inverse (over the entire number line)

Definition 1: One-to-One Function

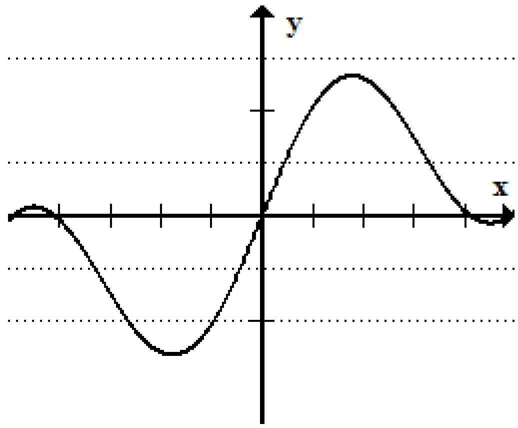
A function is one to one on domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. In other words, a function is one to one if $f(x_1) = f(x_2)$, implies $x_1 = x_2$.

Definition 2: Horizontal Line Test

A function $y = f(x)$ is one to one if and only if its graph intersects each horizontal line at most once. In other words, each y-value comes from only one x-value.



This graph passes the horizontal line test. Pick any horizontal line and you'll find it will cross the graph at most once. So this graph is one-to-one.



This graph fails the horizontal line test. There are multiple places where horizontal lines cross more than once. This graph is not one-to-one.

Definition 3: Inverse Function

Suppose that f is one to one on D with range R. The inverse function f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a$$

Some things to note:

1. Range of f = domain of f^{-1}
2. Domain of f = range of f^{-1}

So what?? If you're ever asked to find the range of a function. Sometimes it's easier to find the inverse function and then find its domain.

Example 1

If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$, find $f^{-1}(7)$, $f^{-1}(5)$, and $f^{-1}(-10)$.

Based on the previous definition, we have

1. $f^{-1}(7) = 3$, because f(3) = 7
2. $f^{-1}(5) = 1$, because f(1) = 5
3. $f^{-1}(-10) = -1$, because f(-1) = -10

Steps 1: Finding the Inverse Function

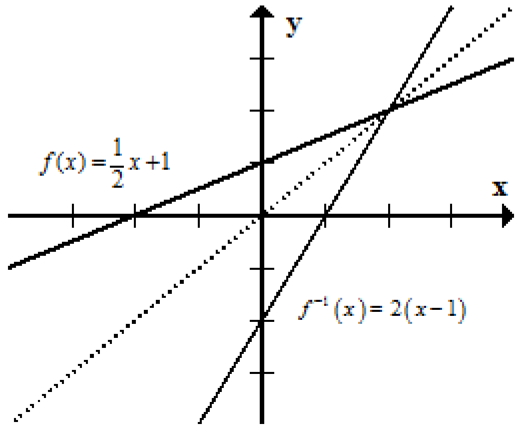
1. Rewrite $f(x)$ as y.
2. Interchange y and x
3. Solve for y. This is $y = f^{-1}(x)$.
4. Note, this does not work for all functions.

Example 2

Find the inverse of $y = \frac{1}{2}x + 1$

1. Swap y and x.
$$x = \frac{1}{2}y + 1$$
2. Solve for y.
$$\begin{aligned} x &= \frac{1}{2}y + 1 \\ x - 1 &= \frac{1}{2}y \\ 2(x-1) &= y \end{aligned}$$
3. Therefore, $f^{-1}(x) = 2(x-1)$

Let's take a look at the graph of $f(x)$ and $f^{-1}(x)$.



The dotted line in the graph is $y = x$. This isn't a coincidence. All functions and their inverses are symmetric over the line $y = x$. This should sense though. We formed the inverse by swapping y for x.

Example 3

Let $f(x) = \sqrt{-x+1} + 2$. Find f^{-1} and graph them both.

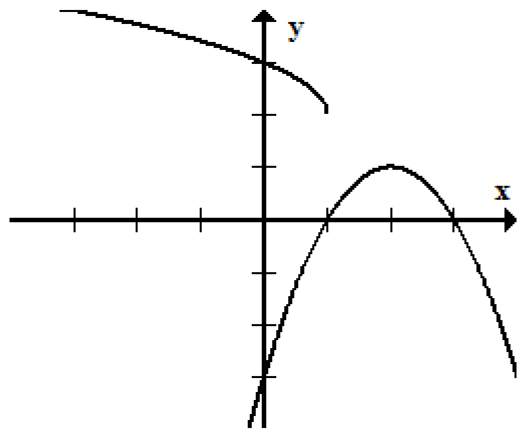
1. We should note the range of $f(x)$. You'll see it in a bit when we look at the graph, but the range of this function is $y \geq 2$.

2. Swap y and x
$$x = \sqrt{-y+1} + 2$$

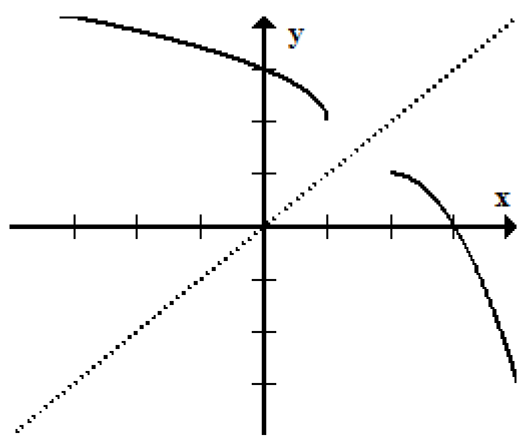
3. Solve for y
$$\begin{aligned} x &= \sqrt{-y+1} + 2 \\ x - 2 &= \sqrt{-y+1} \\ (x-2)^2 &= -y+1 \\ (x-2)^2 - 1 &= -y \\ -(x-2)^2 + 1 &= y \end{aligned}$$

4. Therefore, $f^{-1}(x) = -(x-2)^2 + 1$ line

5. Let's look at the graphs



That doesn't look right. These aren't symmetric. Well recall that the range of $f(x)$ is $y \geq 2$. And since the range of $f(x)$ is the domain of $f^{-1}(x)$, we now know the domain of $f^{-1}(x)$ should be $x \geq 2$. Restricting the domain now we get

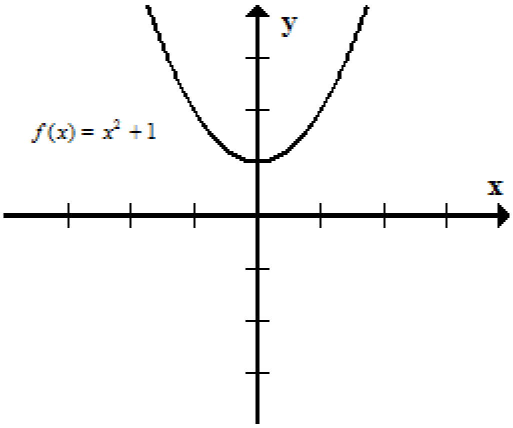


Now they're symmetric. So what does this mean?

It means functions can have inverses but those inverses may have restricted domains.

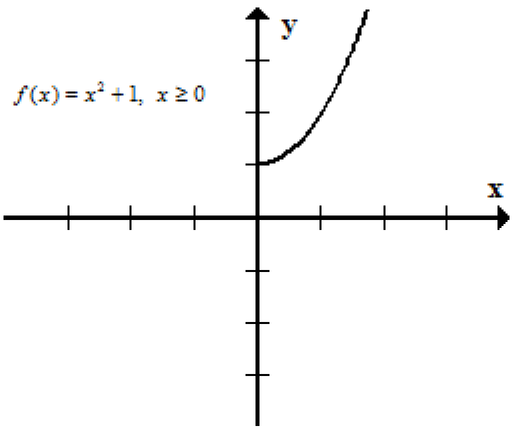
Example 4

Take $f(x) = x^2 + 1$ as another example. If you look at the graph, you see it doesn't pass the horizontal line test.



But can we restrict its domain so it does pass the horizontal test?

Yes, yes we can.



Now it passes the horizontal line test and should have an inverse.

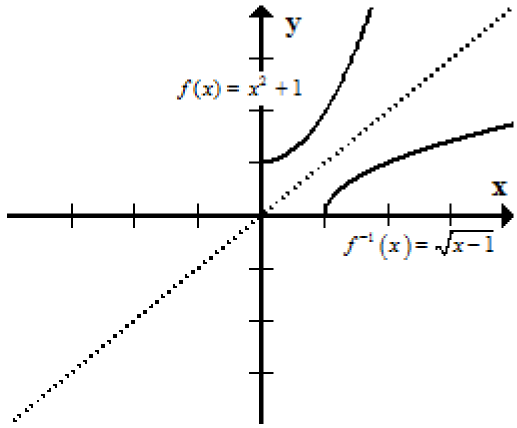
1. Swap y and x

$$y = x^2 + 1 \to x = y^2 + 1$$

2. Solve for y

$$y = \sqrt{x - 1}$$

3. Therefore, $f^{-1}(x) = \sqrt{x-1}$. But remember, we restricted $f(x)$ so its domain is $x \geq 0$ and its range is $y \geq 1$. This means the inverse should have a range $y \geq 0$ and a domain of $x \geq 1$. Is that true?



In case you don't what the function is, you can still work with the inverse. You've seen examples now that show you the inverse is symmetric about the line $y = x$.

Theorem 1

If f is a one to one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

What if I ask you to find the derivative of an inverse function. That could be hard. So we have the following theorem that helps simplify things.

Theorem 2: Derivative of Inverse Theorem

If f is one to one differentiable function with inverse function f^{-1} and $f'\left(f^{-1}(a)\right) \neq 0$, then the inverse function is differentiable at a and

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}$$

Let's call $f^{-1}(x) = g(x)$ to help clean up what's about to come.

The definition of the derivative of $g(a)$ is

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

Since g and f are inverses we also know that

$$g(x) = y \to f(y) = x$$

$$g(a) = b \to f(b) = a$$

$$\text{If } x \to a, \text{ then } y \to b$$

Therefore,

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{y \to b} \frac{y - b}{f(y) - f(b)}$$

A couple quick tricks

$$\begin{aligned} g'(a) &= \lim_{y \to b} \frac{y - b}{f(y) - f(b)} \\ &= \lim_{y \to b} \frac{1}{\frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{\lim_{y \to b} \frac{f(y) - f(b)}{y - b}} \\ &= \frac{1}{f'(b)} \\ &= \frac{1}{f'(g(a))} \end{aligned}$$

But remember that we defined $g(a) = f^{-1}(a)$. So

$$\left(f^{-1}\right)'(a) = \frac{1}{f'(f^{-1}(a))}$$

Example 5

Let $f(x) = x^3 - 2$. Find the value of $\dfrac{df^{-1}}{dx}$ at $x = 6$.

1. Note that $\dfrac{df^{-1}}{dx}$ at $x = 6$ is the same as asking for

$$\left(f^{-1}\right)'(6)$$

2. Our theorem states that

$$\left(f^{-1}\right)'(6) = \frac{1}{f'(f^{-1}(6))}$$

3. So what is $f^{-1}(6)$?

   We're looking for what x value do we plug in to $f(x)$ so $f(x) = 6$.

$$f^{-1}(6) = 2 \text{ because } f(2) = 2^3 - 2 = 6$$

4. So we have $\dfrac{1}{f'(f^{-1}(6))} = \dfrac{1}{f'(2)}$

5. So what's $f'(2)$? Well that one's easy.

$$f'(x) = 3x^2 \text{ so } f'(2) = 12$$

6. Final Answer

$$\begin{aligned}
\left(f^{-1}\right)'(6) &= \frac{1}{f'(f^{-1}(6))} \\
&= \frac{1}{f'(2)} \\
&= \frac{1}{12}
\end{aligned}$$

Example 6

Let $f(x) = 3 + x^2 + \tan(\pi x/2)$, $-1 < x < 1$, and $a = 3$. Find $\left(f^{-1}\right)'(3)$.

1. Use $\left(f^{-1}\right)'(3) = \dfrac{1}{f'\left(f^{-1}(3)\right)}$

2. Find $f^{-1}(3)$:

$$f^{-1}(3) = ? \rightarrow f(?) = 3$$

   To solve this, we set $3 + x^2 + \tan(\pi x/2) = 3$ and solve for x.

   Solving this, we get $x = 0$. Therefore,

$$f^{-1}(3) = 0$$

3. Find $f'(x)$

$$f'(x) = 2x + \sec^2(\pi x/2) \cdot \frac{\pi}{2}$$

4. Evaluate $f'(f^{-1}(3))$:

$$\begin{aligned} f'(f^{-1}(3)) &= f'(0) \\ &= 2(0) + \sec^2(0) \cdot \frac{\pi}{2} \\ &= \frac{\pi}{2} \end{aligned}$$

5. So

$$\begin{aligned} (f^{-1})'(3) &= \frac{1}{f'\left(f^{-1}(3)\right)} \\ &= \frac{1}{\frac{\pi}{2}} \\ &= \frac{2}{\pi} \end{aligned}$$