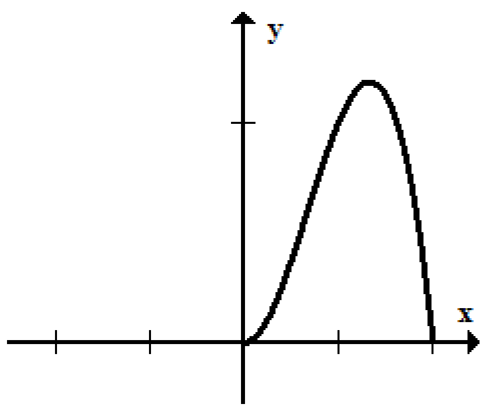
Math 230

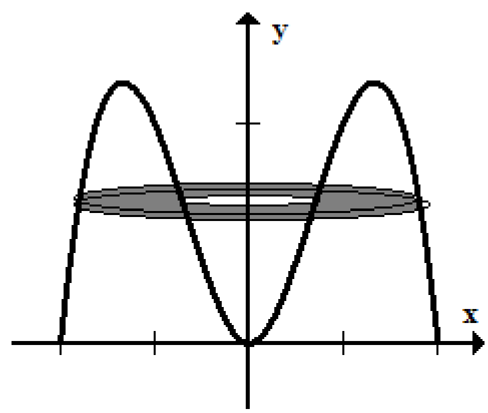
Calculus II

Brian Veitch • Fall 2015 • Northern Illinois University

Volumes by Cylindrical Shells

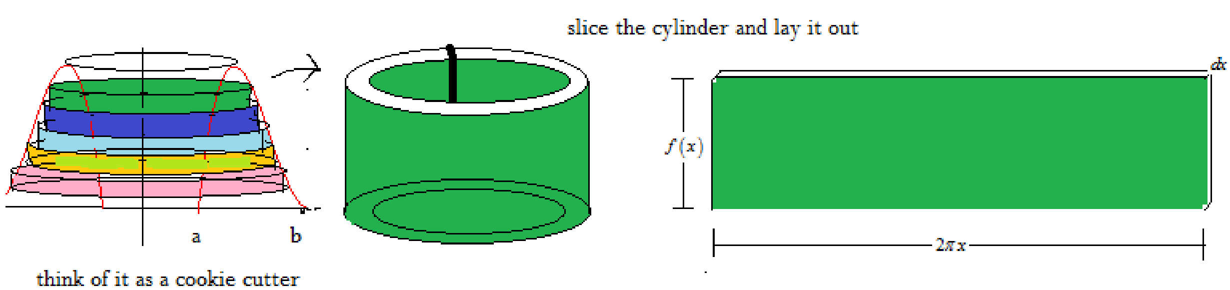
Here's the idea. Take this region and rotate it about the y-axis.





Our main issue in this problem is finding the area of the cross section. To find the area of the washer, we need to find the radius of the outside circle and the radius of the inside circle. The problem is they use the same function. So we need another way of doing this.

We do this by what we call cylindrical shells.



You start slicing and pulling out the cylinders at $x = a$ and stop at $x = b$. Note that the radius of the cylinder is the distance from the axis of revolution to the outside of the cylinder. In this case, that distance is x. The area of the front of the cylinder is

$$A(x) = 2\pi x \cdot f(x)$$

.

The general formula for the area is

$$2\pi(radius)(height)$$

The volume is then

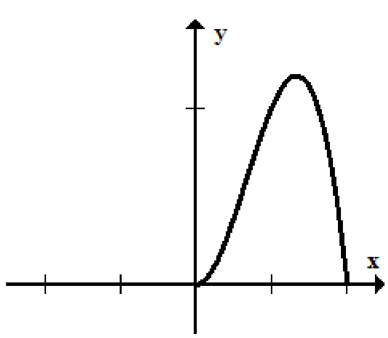
$$V = \int_a^b 2\pi x \cdot f(x)\, dx$$

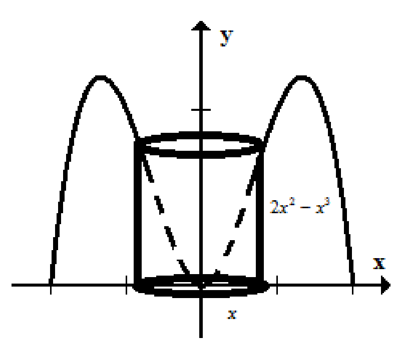
A quick note: You always integrate about the axis you're slicing. In this problem, we are slicing along the x-axis to make our cylinders. Therefore, everything should be in terms of x (even though we rotated about the y-axis.

Example 1

Find the volume of the solid obtained by rotating the region bounded by $y = 2x^2 - x^3$ and $y = 0$ around the y-axis.

1. Draw a picture and a typical cylinder





Always ask yourself, what is the radius? Where is the center of the cylinder? These help you figure out the function $R(x)$.

(a) The distance from the y-axis to the outside of the cylinder is x.

(b) The cylinder height is $f(x) = 2x^2 - x^3$

2. Find the area:

$$A(x) = 2\pi(radius)(height) = 2\pi x \cdot (2x^2 - x^3)$$

$$A(x) = 2\pi(2x^3 - x^4)$$

3. What are the limits of integration?

We begin slicing at $x = 0$ and stop at $x = 2$.

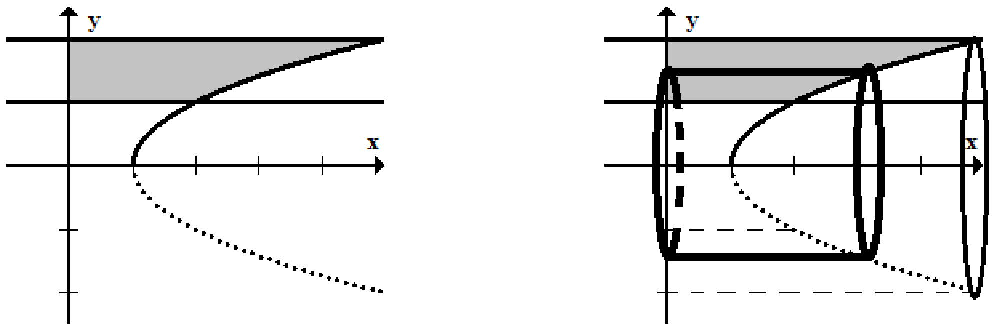
4. Find the volume and evaluate

$$\begin{aligned} V &= \int_0^2 2\pi(2x^3 - x^4)\, dx \\ &= 2\pi \int_9^2 2x^3 - x^4\, dx \\ &= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}\Big|_0^2 \right] \\ &= 2\pi \left(8 - \frac{32}{5} \right) - 0 \\ &= \frac{16}{5}\pi \end{aligned}$$

Example 2

Find the volume of the the region bounded by $x = 1 + y^2$, $x = 0$, $y = 1$, and $y = 2$ rotated around the x-axis.

1. Draw a picture and a typical cylinder



NOTE: The cylinders are created by slicing alone the y-axis. Everything must be in terms of y.

2. Find the radius and height of the cylinder

   (a) The radius is the distance from the axis of revolution (x-axix) to the outside of the cylinder. In this case, it's just y.

$$R(y) = y$$

   (b) The height of the cylinder is the distance from the y-axis to the top of the cylinder. That height is $x = 1 + y^2$, i.e., $h = 1 + y^2$.

$$H(y) = 1 + y^2$$

3. Find the area, $A(y)$.

$$A(y) = 2\pi(radius)(height) = 2\pi y(1 + y^2)$$

4. Find the limits of integration.

You start creating the cylinders at $y = 1$ and end at $y = 2$.

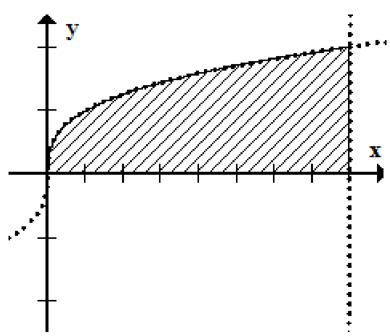
5. Find the volume and evaluate

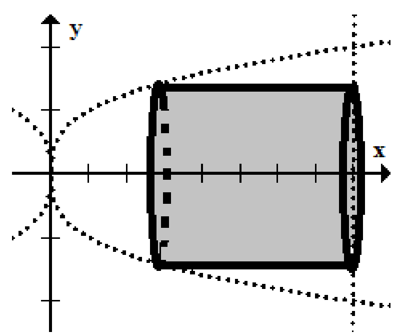
$$
\begin{aligned}
V &= \int_1^2 2\pi y(1+y^2)\,dy \\
&= 2\pi \int_1^2 y^3 + y\,dy \\
&= 2\pi \left[\frac{1}{4}y^4 + \frac{1}{2}y^2 \Big|_1^2 \right] \\
&= 2\pi \left(6 - \frac{3}{4} \right) \\
&= \frac{21}{2}\pi
\end{aligned}
$$

Example 3

Determine the volume of the solid obtained by rotating the region bounded by y = $\sqrt[3]{x}$, $y = 0$ and $x = 8$.

1. Start by drawing the pictures





Note that we can do this by cylinders or washers (disks).

2. Find the radius and height. Since we're slicing along the y axis to create our cylinders, everything must be in terms of y.

(a) The radius of the cylinder is the distance from the x-axis the curve $y = \sqrt[3]{x}$, which is just y.

$$R(y) = y$$

(b) The height of the cylinder is $8 - y^3$

3. Find the area, $A(y)$

$$A(y) = 2\pi(radius)(height) = 2\pi(y)(8 - y^3)$$

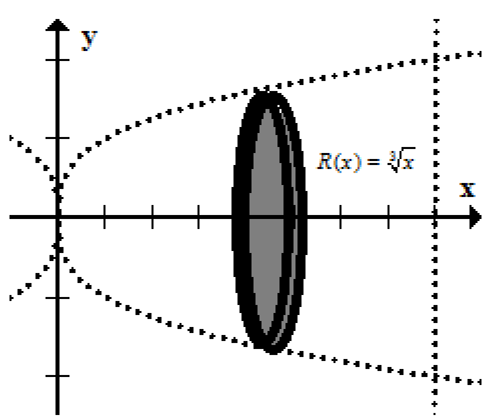
4. Find the limits of integration

We start creating the cylinders at $y = 0$ and end at $y = 2$.

5. Find the volume V and evaluate

$$\begin{aligned} V &= \int_0^2 2\pi y(8 - y^3)\, dy \\ &= 2\pi \int_0^2 8y - y^4\, dy \\ &= 2\pi \left[4y^2 - \frac{1}{5}y^5 \Big|_0^2 \right] \\ &= \frac{96}{5}\pi \end{aligned}$$

If you were to do this problem using the method of disks, you'd get

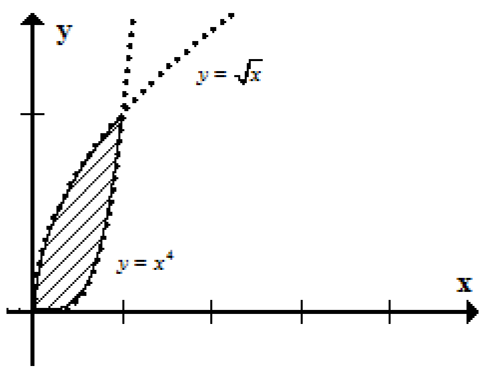


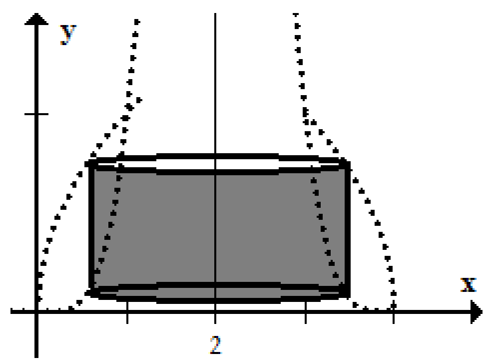
$$
\begin{aligned}
V &= \int_0^8 \pi (R(x))^2 \, dx \\
&= \int_0^8 \pi (x^{1/3})^2 \, dx \\
&= \int_0^8 \pi x^{2/3} \, dx \\
&= \frac{3}{5}\pi x^{5/3} \Big|_0^8 \\
&= \frac{96}{5}\pi
\end{aligned}
$$

Example 4

Find the volume of the region bounded by $y = x^4$ and $y = \sqrt{x}$ about the line $x = 2$.

1. Draw a picture and a typical cylinder.





Even though it is possible to do this problem using washers, you'd find it difficult (but not impossible). You might as well try it on your own to see.

2. Find the radius and height of a cylinder.

(a) The radius is the distance from the axis of revolution ($x = 2$) to the edge of the cylinder. Since this has to be in terms of x, we write $2 - x$.

(b) The height of the cylinder is just

$$H(x) = \sqrt{x} - x^4$$

3. Find the area, $A(x)$

$$A(x) = 2\pi(radius)(height) = 2\pi(2-x)(x^{1/2} - x^4)$$

4. Find the limits of integration.

We start slicing and creating cylinders at $x = 0$ and stop at $x = 1$.

5. Find V and evaluate.

$$\begin{aligned} V &= \int_0^1 2\pi(2-x)(x^{1/2} - x^4)\,dx \\ &= 2\pi \int_0^1 x^5 - 2x^4 - x^{3/2} + 2x^{1/2}\,dx \\ &= 2\pi \left[\frac{1}{6}x^6 - \frac{2}{5}x^5 - \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} \Big|_0^1 \right] \\ &= \frac{7}{5}\pi \end{aligned}$$