

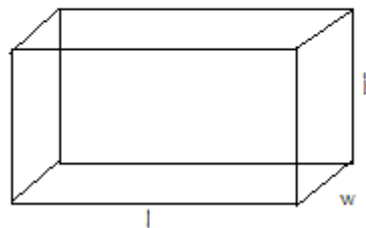
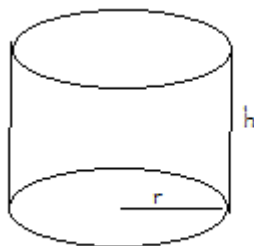
# MATH 230

## CALCULUS II

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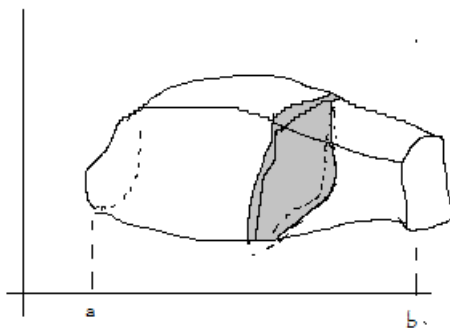
### Volumes by Washer Method

We know how to find volumes of well-established solids such as a cylinder or rectangular box.



What happens when the volume can't be found quite as easily nice or when it doesn't have a nice formula like the previous two?

Consider the following,

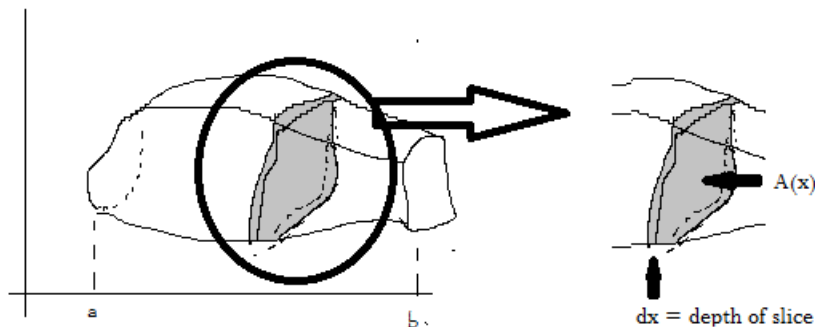


The idea is much like finding the volume of a loaf a break. You take a piece out from the solid (loaf of bread) and find the volume of that small piece. You keep slicing the solid and finding the volume of each slice. You then add up all the slides to find the estimated

volume of the solid.

$$V = \int_a^b A(x) dx$$

where  $A(x)$  is the area of the slice  $dx$  is its depth.



### Steps 1: General Steps to Find the Volume of a Solid

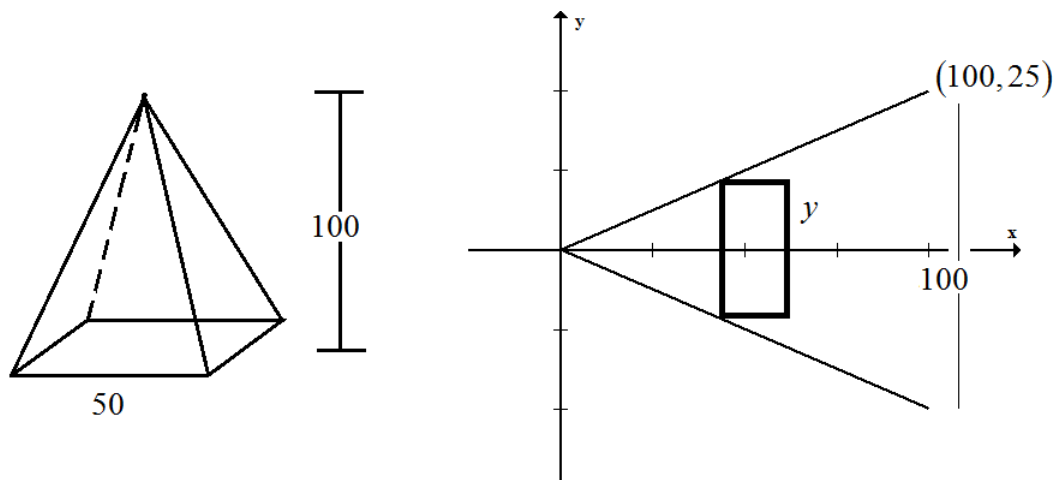
1. Sketch the solid and a typical cross section. The typical cross section is **extremely** useful.
2. Find a formula for  $A(x)$ , the area of a typical cross section.
3. Find the limits of integration. At what value of  $x$  does the solid begin and at what value of  $x$  does the solid end?
4. Integrate  $A(x)$  from  $a$  to  $b$

$$V = \int_a^b A(x) dx$$

### Example 1

Find the volume of a square base pyramid that has a base of 50 and height of 100.

1. First, let's draw a picture to see what we have.



- If we left the side of the square slice by  $2y$ , then the area of the cross section is  $A(x) = (2y)^2$ .
- We need to find a formula for  $y$  in terms of  $x$ . Since the side of the pyramid is a straight line starting at the origin, it must have the form  $y = mx$ .
- Let's find  $m$ .

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{25 - 0}{100 - 0} = \frac{1}{4}$$

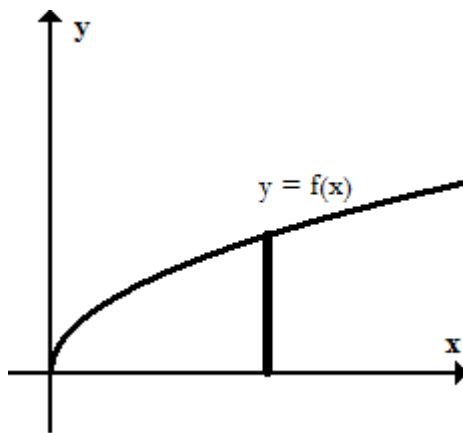
So the equation of the side of the cross section is  $2y = 2 \left( \frac{1}{4}x \right) = \frac{1}{2}x$ .

- Bounds? Ask yourself the following question, "where do you start the slicing on the  $x$ -axis and where do you stop the slicing?" We start slicing at  $x = 0$  and end slicing at  $x = 100$ .
- Putting this all together

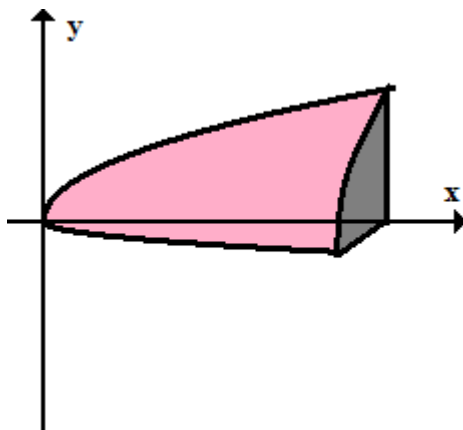
$$\begin{aligned} V &= \int_a^b A(x) dx \\ &= \int_0^{100} \left( \frac{1}{2}x \right)^2 dx \\ &= \int_0^{100} \frac{1}{4}x^2 dx \\ &= \frac{1}{12}x^3 \Big|_0^{100} \\ &\approx 83,333.33\dots \end{aligned}$$

## Method of Disks

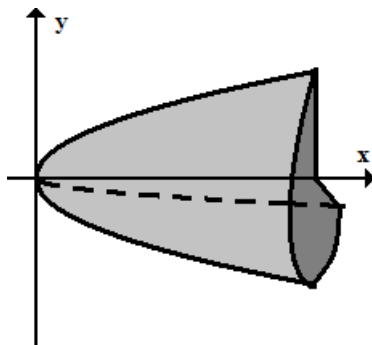
1. Define a function  $y = f(x)$  over an interval  $[a,b]$ . The function is usually given to you (but not always).
2. Rotate the region about the  $x$ -axis to get the surface of the solid of revolution.
  - (a) Start with the given function  $y = f(x)$ .



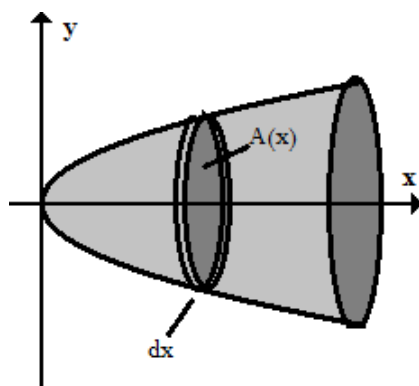
- (b) Then you'd begin by rotating that area around the  $x$ -axis.



- (c) Keep going...



(d) A little bit more...



3. Now that you have your solid of revolution, you have to find the cross-sectional area  $A(x)$ . Did you notice that the cross sectional area is a circle? So  $A(x) = \pi r^2$

And what's the radius of the this circle? It's  $f(x)$ , the distance from the  $x$ -axis to the outside of the solid. You can see why if you go back to the first graph right after 2(a). So now,

$$A(x) = \pi(f(x))^2$$

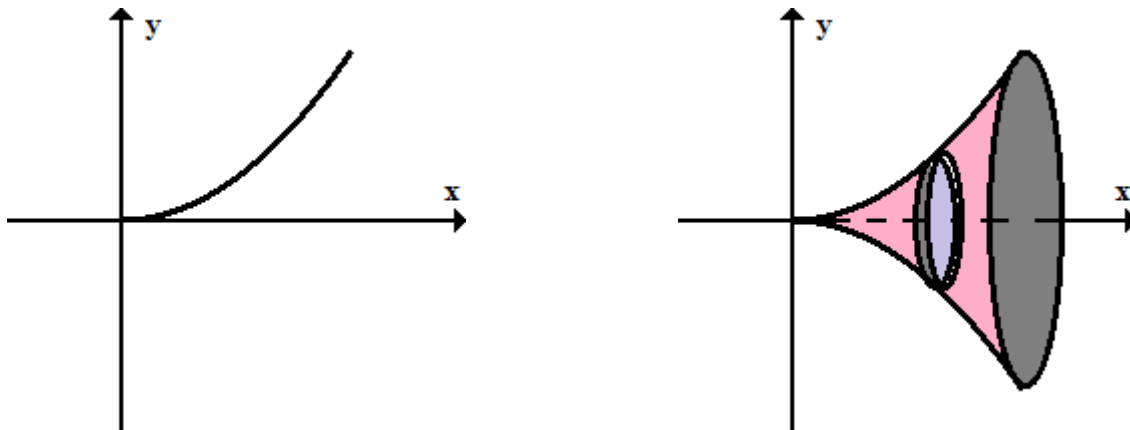
4. Since you have to add up all these circles from  $x = a$  to  $x = b$ , we write

$$V = \int_a^b (R(x))^2 dx$$

**Example 2**

The region under the graph of  $y = x^2$  over the interval  $[0,3]$  is rotated about the  $x$ -axis. Find the volume of the solid.

1. Let's draw a picture of the area and then another after it's been rotated.



2. Find  $A(x)$ , the area of a typical cross section.

The cross section is a circle with radius  $y = x^2$ . Therefore,

$$A(x) = \pi r^2 = \pi(x^2)^2 = \pi x^4$$

3. The volume formula for this problem is

$$\begin{aligned} V &= \int_0^3 A(x) \, dx \\ &= \int_0^3 \pi x^4 \, dx \\ &= \left. \frac{1}{5} \pi x^5 \right|_0^3 \\ &= \frac{32\pi}{5} \end{aligned}$$

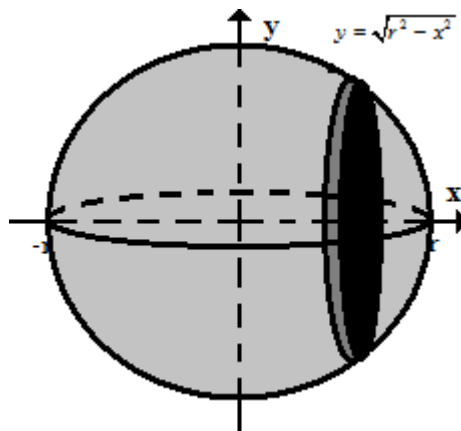
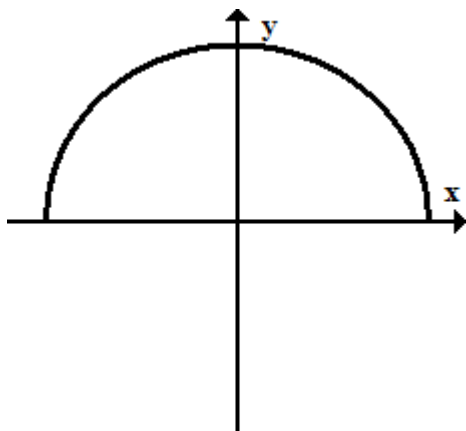
**Example 3**

Find the volume of a sphere of radius  $r$ .

1. We need a function to rotate about the  $x$ -axis that will form the sphere. We will use a semi-circle with radius  $r$ .

The equation for this semi-circle is  $x^2 + y^2 = r^2$ . Solving for  $y$  we get

$$y = \sqrt{r^2 - x^2}$$



2. Find  $A(x)$ , the area of the cross-section.

Since the radius is  $y = \sqrt{r^2 - x^2}$ ,

$$A(x) = \pi \left( \sqrt{r^2 - x^2} \right)^2 = \pi(r^2 - x^2)$$

3. Find the volume

$$\begin{aligned}
 V &= \int_{-r}^r A(x) \, dx \\
 &= \int_{-r}^r \pi(r^2 - x^2) \, dx \\
 &= \int_{-r}^r \pi r^2 - \pi x^2 \, dx \\
 &= \left. \pi r^2 x - \frac{\pi}{3} x^3 \right|_{-r}^r \\
 &= \left( \pi r^2 \cdot r - \frac{\pi}{3} r^3 \right) - \left( \pi r^2 \cdot (-r) - \frac{\pi}{3} (-r^3) \right) \\
 &= 2\pi r^3 - \frac{2\pi}{3} r^3 \\
 &= \frac{4}{3} \pi r^3
 \end{aligned}$$

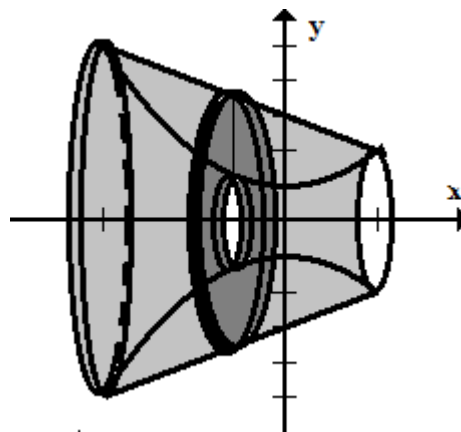
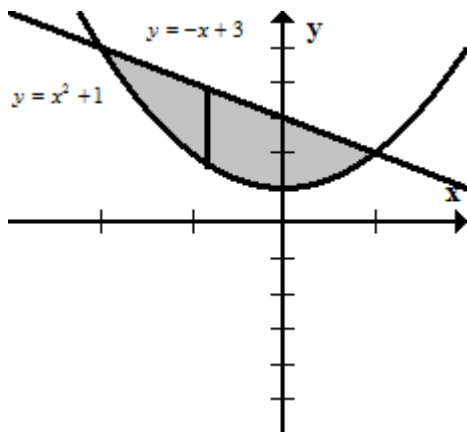
So that's where this formula comes from!

## Method of Washers

### Example 4

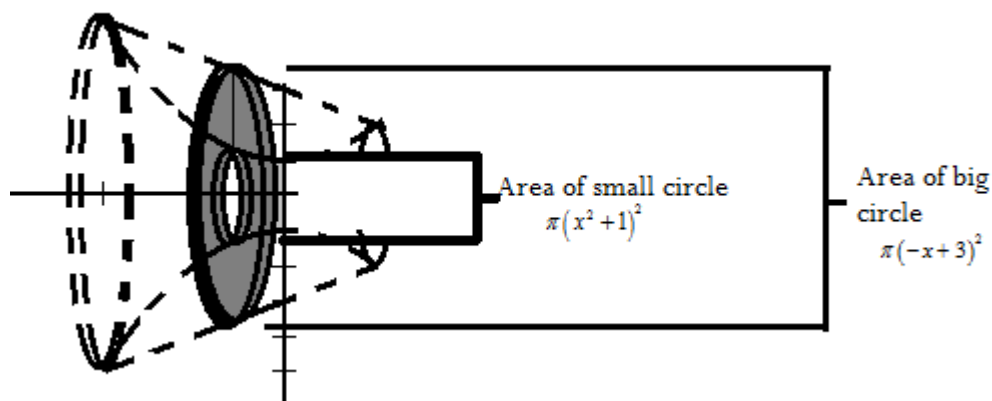
Revolve the area bounded by the curve  $y = x^2 + 1$  and the line  $y = -x + 3$ . Compute the volume of the resulting solid.

1. As always, let's draw a picture.





Let's remove the washer so we can examine more,



2. Find  $A(x)$

Finding the area of the cross section is like finding the area between curves. The area of the washer is the area of the outside circle minus the area of the inside circle.

A good way to do this is to find the radius of the outside and inside circles.

$$R(x) = -x + 3$$

and

$$r(x) = x^2 + 1$$

The area of the washer is

$$A(x) = \pi [(-x + 3)^2] - [\pi (x^2 + 1)^2]$$

3. Find the limits of integration. In other words, at what  $x$ -value do we begin the volume and at what  $x$ -value do we end the volume. To find these, we find the intersection of the two functions.

$$\begin{aligned} -x + 3 &= x^2 + 1 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \end{aligned}$$

So the limits of integration are  $x = -2$  to  $x = 1$ .

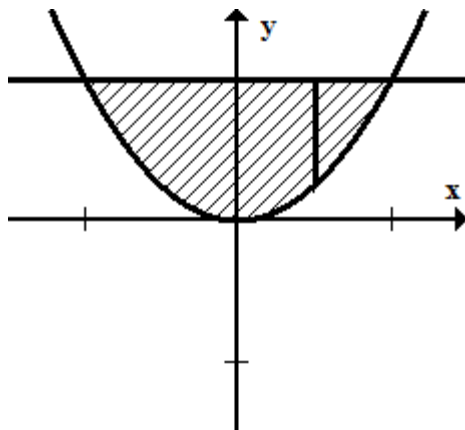
4. Set up the integral for  $V(x)$ .

$$\begin{aligned} V &= \int_{-2}^1 \pi(R(x))^2 - \pi(r(x))^2 dx \\ &= \int_{-2}^1 [\pi(-x + 3)^2] - \pi[(x^2 + 1)^2] dx \\ &= \pi \int_{-2}^1 -x^4 - x^2 - 6x + 8 dx \\ &= \pi \left[ -\frac{1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x \right]_{-2}^1 \\ &= 23.4\pi \text{ (trust me)} \end{aligned}$$

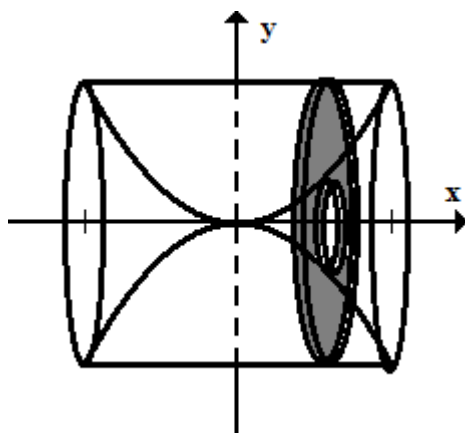
### Example 5

The region bounded by  $y = x^2$  and  $y = 1$  is revolved about the  $x$ -axis. Find the volume of the solid.

1. Let's begin by drawing a picture of the shaded region.



2. Now rotate it around the  $x$ -axis while also creating the typical cross section.



3. To find the area of the cross section, we identify the outside radius and the inside radius.

$R(x) = 1$ , note: distance from  $x$ -axis to the outside of the solid is always 1

$$r(x) = x^2$$

4. Find  $A(x)$

$$\begin{aligned}A(x) &= \pi(R(x))^2 - \pi(r(x))^2 \\&= \pi(1)^2 - \pi(x^2)^2 \\&= \pi - \pi x^4 \\&= \pi(1 - x^4)\end{aligned}$$

5. Find the limits of integration.

It's not too hard to find these. From the graph, we see they intersect at  $x = -1$  and  $x = 1$

6. Find  $V$  and evaluate

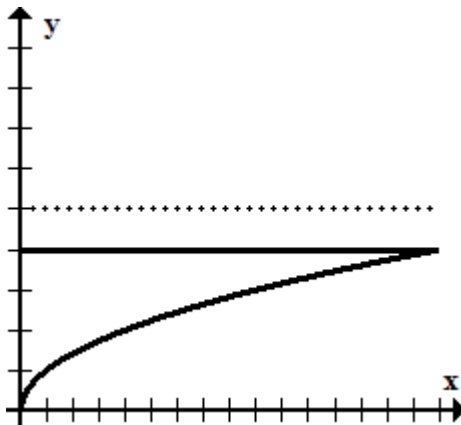
$$\begin{aligned}V &= \int_{-1}^1 A(x) dx \\&= \int_{-1}^1 \pi(1 - x^4) dx \\&= \pi \left[ x - \frac{1}{5}x^5 \right]_{-1}^1 \\&= \frac{8\pi}{5}\end{aligned}$$

So far all the solids were obtained by rotating the area around the  $x$ -axis. But the  $x$ -axis is just one horizontal line. Can we rotate an area around other horizontal lines? You bet!

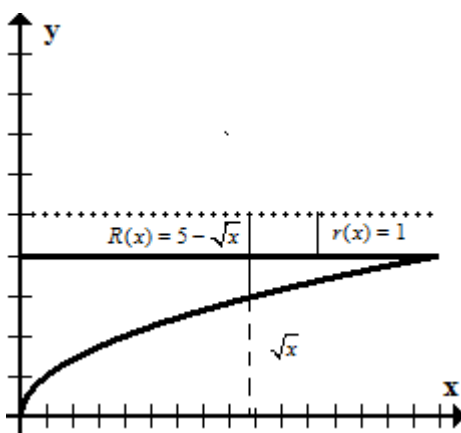
### Example 6

Find the volume of the solid obtained by revolving the region enclosed by  $y = \sqrt{x}$ ,  $y = 4$ ,  $x = 0$  about the horizontal line  $y = 5$ .

- Let's find the shaded region that we're rotating.



I'd like to remind you that the radius of our washers (or disks) is always the distance from the **axis of revolution** to the edge of the circle. We can actually find  $R(x)$  and  $r(x)$  now even before rotating.



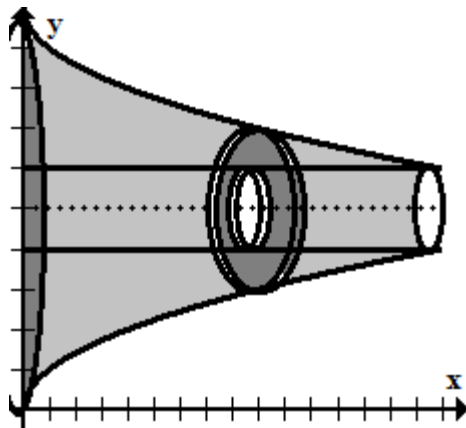
From the graph, you can see

$$R(x) = 5 - \sqrt{x}$$

and

$$r(x) = 1$$

- We can actually move on to finding  $A(x)$ , but let's go ahead and draw the solid for fun.



3. Fun? I know, right!. Ok, now let's find  $A(x)$

$$\begin{aligned}
 A(x) &= \pi(R(x))^2 - \pi(r(x))^2 \\
 &= \pi(5 - \sqrt{x})^2 - \pi(1)^2 \\
 &= \pi(25 - 10\sqrt{x} + x - 1) \\
 &= \pi(24 - 10x^{1/2} + x)
 \end{aligned}$$

4. The limits of integration are from  $x = 0$  to  $x = 16$ .

5. Find  $V$  and evaluate

$$\begin{aligned}
 V &= \int_0^{16} A(x) dx \\
 &= \pi \int_0^{16} 24 - 10x^{1/2} + x dx \\
 &= \pi \left[ 24x - \frac{20}{3}x^{3/2} + \frac{1}{2}x^2 \right]_0^{16} \\
 &= 85.3333\pi
 \end{aligned}$$

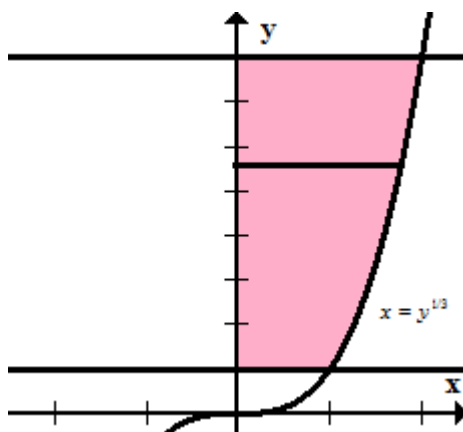
**All right then!** We've done enough examples of rotating areas around the  $x$ -axis. Now we do it **all over again** by rotating around the  $y$  axis. The idea is still the same. Find the

radius, find the area of cross section, integrate over some interval.

### Example 7

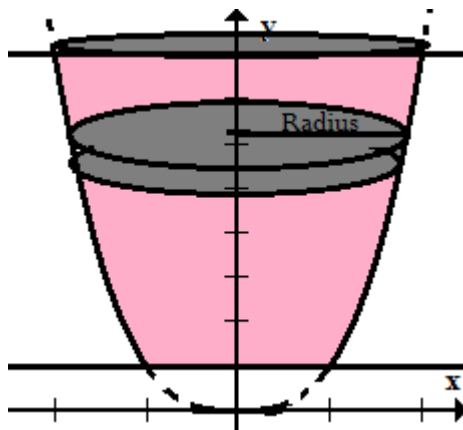
Find the volume of the solid obtained by rotating the region bounded  $y = x^3$ ,  $y = 8$ ,  $y = 1$  and  $x = 0$  about the  $y$  axis,  $x$ -axis,  $x = 2$ ,  $x = -2$ .

1. Draw a picture



Notice that I wrote  $x = y^{1/3}$ . That's because everything has to be in terms of  $y$ . So the radius has to be of the form  $x = f(y)$ .

2. Rotate about the  $y$ -axis.



3. Find the radius of the cross section (in terms of  $y$ )

$$R(y) = y^{1/3}$$

4. Find  $A(y)$

$$A(x) = \pi (x^{1/3})^2 = \pi x^{2/3}$$

5. Find the limits of integration

Again, since everything is in terms of  $y$ , the limits do as well. At what  $y$ -value does the volume start and at what  $y$ -value does it end?

$$y = 1 \text{ and } y = 8$$

6. Find  $V$  and evaluate

$$\begin{aligned} V &= \int_1^8 \pi A(y) \, dy \\ &= \int_1^8 \pi y^{2/3} \, dy \\ &= \left. \frac{3}{5} \pi y^{5/3} \right|_1^8 \, dy \\ &= \frac{3}{5} \pi (8)^{5/3} - \frac{3}{5} \pi \\ &= 18.6\pi \end{aligned}$$

I'm just going to set up the other rotations

$$(a) \text{ } x\text{-axis: } \int_0^1 \pi(8)^2 - \pi(1)^2 \, dx + \int_1^2 \pi(8)^2 - \pi(x^3)^2 \, dx$$

$$(b) \text{ } x = -2: \int_0^2 \pi(2 + y^{1/3})^2 - \pi(2)^2 \, dy$$

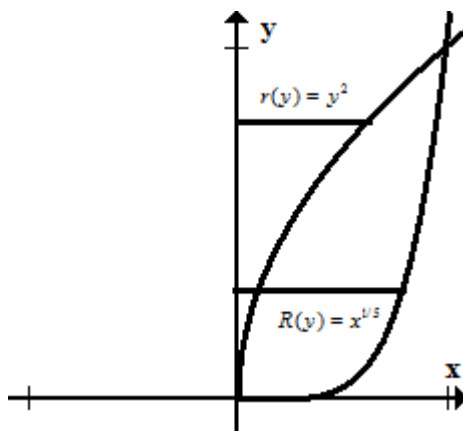
$$(c) \text{ } x = 2: \int_0^2 \pi(2)^2 - \pi(2 - y^{1/2})^2 \, dy$$



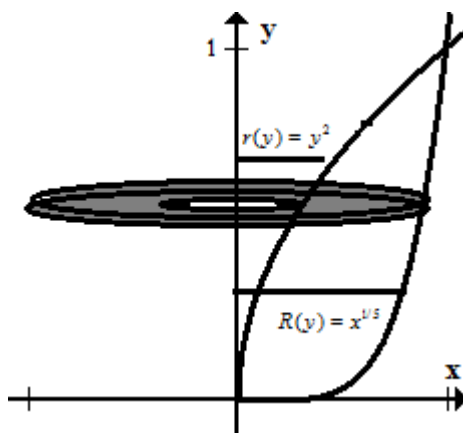
**Example 8**

Revolve the region bounded by  $y = \sqrt{x}$  and  $y = x^5$  about the  $y$ -axis. Find the volume of the solid.

1. Draw a picture



Inside of rotating the entire region, I just want to rotate the cross-sectional piece.



2. Identify the outside and inside radii

$$R(y) = y^{1/5}$$

$$r(y) = y^2$$

3. Find  $A(y)$

$$\begin{aligned} A(y) &= \pi(R(y))^2 - \pi(r(y))^2 \\ &= \pi(y^{1/5})^2 - \pi(y^2)^2 \\ &= \pi y^{2/5} - \pi y^4 \end{aligned}$$

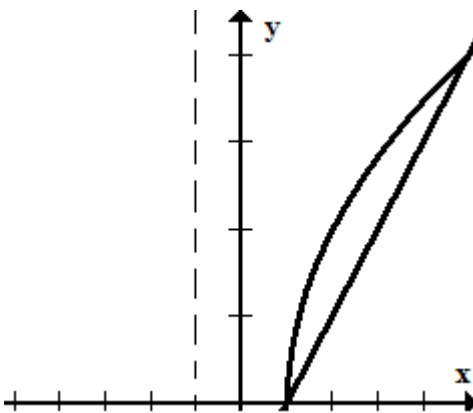
4. Find  $V$  and evaluate

$$\begin{aligned} V &= \int_0^1 A(y) \, dy \\ &= \int_0^1 \pi y^{2/5} - \pi y^4 \, dy \\ &= \left. \frac{5}{7} \pi y^{7/5} - \frac{1}{5} \pi y^5 \right|_0^1 \\ &= \frac{18}{35} \pi \end{aligned}$$

### Example 9

Determine the volume of the solid obtained by rotating the region bounded by  $y = 2\sqrt{x-1}$  and  $y = x-1$  about the  $x = -1$ .

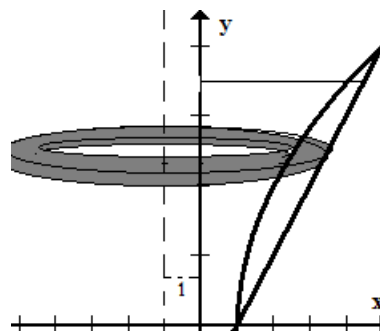
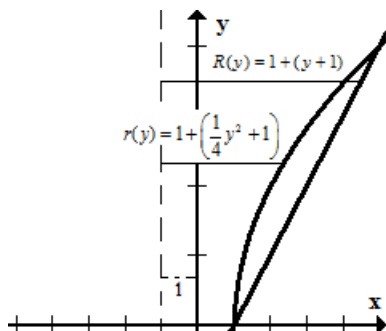
1. Draw a picture



2. Make sure you write all your functions in terms of  $y$ .

$$y = 2\sqrt{x-1} \rightarrow x = \frac{1}{4}y^2 + 1$$

$$y = x - 1 \rightarrow x = y + 1$$



3. Identify  $R(y)$  and  $r(y)$ .

Recall that the radius will be the distance from the **axis of revolution** to the outside of the circle. Since the the axis of revolution was shifted to the left 1 unit, we have to add 1 to our radii.

$$R(y) = 1 + (y + 1) = y + 2$$

$$r(y) = 1 + \left(\frac{1}{4}y^2 + 1\right) = \frac{1}{4}y^2 + 2$$

4. Find the area of the cross section

$$\begin{aligned} A(y) &= \pi (R(y)^2 - r(y)^2) \\ &= \pi \left( (y + 2)^2 - \left(\frac{1}{4}y^2 + 2\right)^2 \right) \\ &= \pi \left( 4y - \frac{1}{16}y^4 \right) \end{aligned}$$

5. Find the limits of integration

Setting  $\frac{1}{4}y^2 + 1 = y + 1$  gets us  $y = 0$  and  $y = 4$ .

6. Find  $V$  and evaluate

$$\begin{aligned} V &= \int_0^4 A(y) dy \\ &= \pi \int_0^4 4y - \frac{1}{16}y^4 dy \\ &= \pi \left[ 2y^2 - \frac{1}{80}y^5 \right]_0^4 \\ &= \frac{96}{5}\pi \end{aligned}$$