Math 230

Calculus II

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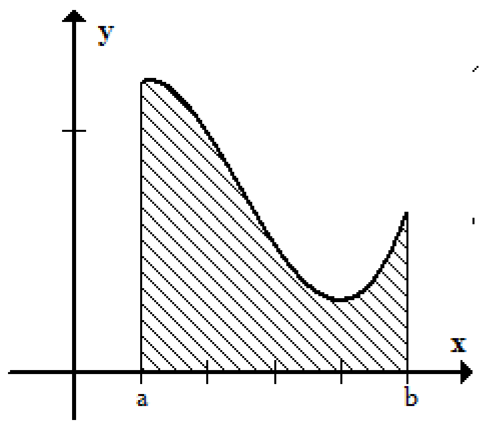
Area between Curves

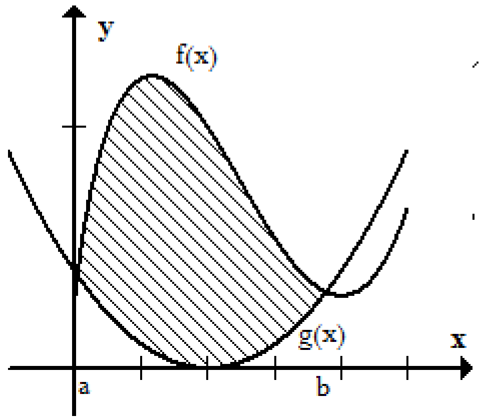
In this section we are going to find the area between curves. Recall that the integral can represent the area between $f(x)$ and the x-axis. And any area below the x-axis is considered negative.



Formula 1: Area under $f(x)$ where $f(x) \geq 0$

$$A = \int_a^b f(x)\, dx$$

So the next question is, how do I find the area of the shaded region below?



Formula 2

Notice that $f(x) \geq g(x)$ on the interval [a,b]. The formula for the area between $f(x)$ and $g(x)$ is

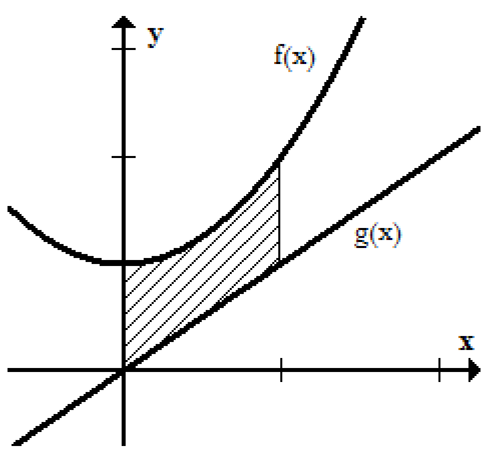
$$\int_a^b f(x) - g(x)\, dx$$

This should make sense. You find the area below $f(x)$ and subtract the area under $g(x)$, which leaves just the area between the two functions.

Example 1

Find the area of the region bounded by $f(x) = x^2 + 1$, $g(x) = x$, $x = 0$, and $x = 1$.

1. The first step would be to sketch and find the shaded region. We need to know which function is on top and which is on the bottom.



2. Now that we see $f(x) = x^2 + 1$ is on top, we can set up our integral.

$$\int_0^1 (x^2+1) - x \, dx$$

3. Now evaluate it,

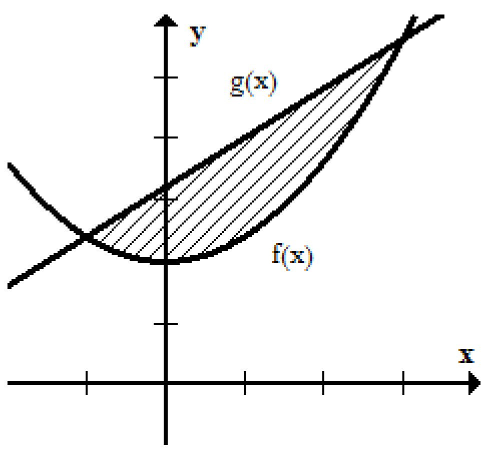
$$\begin{aligned} \int_0^1 (x^2+1) - x \, dx &= \int_0^1 x^2 - x + 1 \, dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x \Big|_0^1 \\ &= \frac{5}{6} - 0 \\ &= \frac{5}{6} \end{aligned}$$

Example 2

Find the area of the region bounded by $f(x) = 2x^2 + 10$ and $g(x) = 4x + 16$

We begin by sketching the graph. Notice that we don't have any bounds and we don't know which function is on top.

1. The sketch is below



2. You might be able to see that the two functions intersect at $x = -1$ and $x = 3$. I want to verify this by doing some algebra. If we want to know where the two functions

intersect, then set them equal to each other.

$$\begin{aligned} 2x^2+10 &= 4x+16 \\ 2x^2-4x-6 &= 0 \\ 2(x^2-2x-3) &= 0 \\ 2(x-3)(x+1) &= 0 \end{aligned}$$

Solving for x and we find $x=-1$ and $x=3$.

3. We can see $g(x)$ is above $f(x)$. If you didn't have a graph to verify that, then just plug in any point between $x=-1$ and $x=3$. Whichever has the larger y-value must be the top function.

4. Set up the integral and evaluate

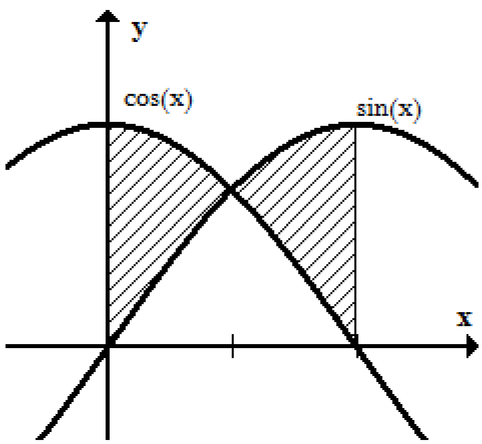
$$\begin{aligned} A &= \int_{-1}^{3} \left(\overset{top}{function}\right) - \left(\overset{bottom}{function}\right)\, dx \\ &= \int_{-1}^{3} (4x+16)-(2x^2+10)\, dx \\ &= \int_{-1}^{3} -2x^2+4x+6\, dx \\ &= -\frac{2}{3}x^3+2x^2+6x\Big|_{-1}^{3} \\ &= 18-\left(\frac{-10}{3}\right) \\ &= \frac{64}{3} \end{aligned}$$

What happens when $f(x) \geq g(x)$ is not always true in the interval [a,b]? Let's take a look at the following example.

Example 3

Find the area of the region bounded by the curves $f(x) = \sin(x)$, $g(x) = \cos(x)$, $x = 0$, and $x = \pi/2$.

1. Let's begin by sketching the graph



   Do you see that $f(x) = \sin(x)$ and $g(x) = \cos(x)$ switch being the top function somewhere in our interval $[0, \pi/2]$. So how do we find out where they switch?

   We need to find the intersection. So set $\sin(x) = \cos(x)$ and solve for x. Solving this gets us $x = \pi/4$.

2. Now, we break the integral up so that each integral has the form **top function - bottom function** in its interval.

$$A = \int_0^{\pi/4} \cos(x) - \sin(x)\, dx \quad + \quad \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\, dx$$

3. Integrate and Evaluate

$$\begin{aligned}
A &= \int_0^{\pi/4} \cos(x) - \sin(x)\, dx + \int_{\pi/4}^{\pi/2} \sin(x) - \cos(x)\, dx \\
&= \left[\sin(x) + \cos(x)|_0^{\pi/4}\right] + \left[-\cos(x) - \sin(x)|_{\pi/4}^{\pi/2}\right] \\
&= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (1)\right] + \left[(-1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] \\
&= 2\sqrt{2} - 2
\end{aligned}$$

Steps for Area Between Functions of x

Steps 1

1. Sketch the graph (if anything, just to see what it looks like)
2. Identify the top and bottom function.

   If they switch, find out where they switch, by setting the functions equal to each other, and then separate the integral.
3. Find the limits of integration. They may tell you this, or you have to find them. Looking at the graph would help very much.
4. Set up the integral and evaluate

$$A = \int_a^b \left(\overset{top}{function}\right) - \left(\overset{bottom}{function}\right)\, dx$$

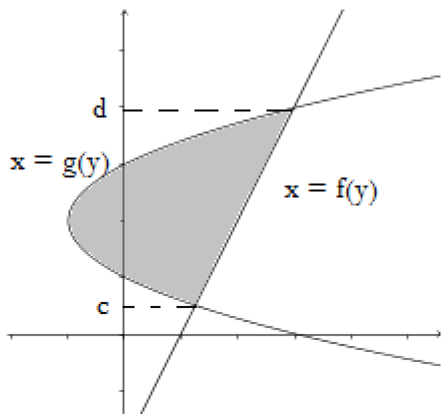
Finding the Area Between Functions of y

We're moving on to a variation of finding the area between curves. All the functions we've dealt with so far have been in terms of x. Now we have to find the area between curves when they are in terms of y.

Formula 3: Formula for the area between curves

$$A = \int_c^d \left(\overset{right}{function}\right) - \left(\overset{left}{function}\right)\, dy$$

$$A = \int_c^d f(y) - g(y)\, dy$$

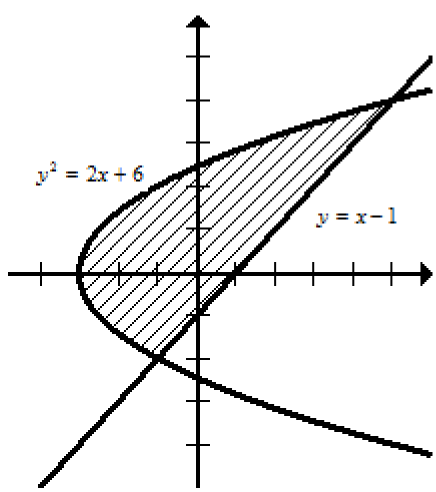


Since your functions are in terms of y, we integrate along the y-axis, from c to d.

Example 4

Find the area enclosed by the line $y = x - 1$ and $y^2 = 2x + 6$.

So why can't we just solve this like the others? Well for one thing, $y^2 = 2x + 6$ is not in the form $y = f(x)$. Let's take a look at the graph.



Notice that there are parts of the graph where $y^2 = 2x + 6$ is both the top and bottom function. That is why it's not a good idea to try integrating with respect to x. But actually, I'm getting a bit ahead of myself. Graphing calculators don't graph in terms of y. So how did I graph this?

Graphing Functions in terms of y

1. If you need to graph in terms of y, set up your equations in the form $x = f(y)$.

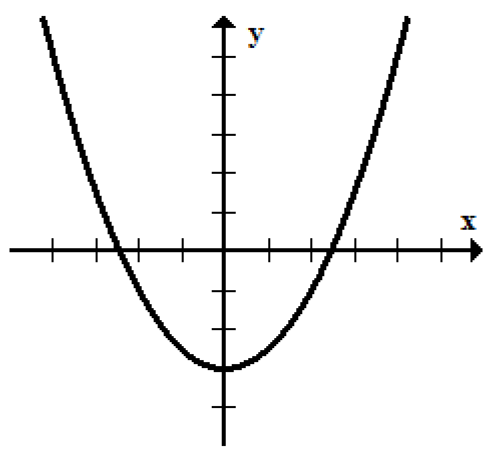
   IMPORTANT: If you also have functions that are in terms of x and can be easily graphed, wait until after you finish graphing the functions in terms of y.

   Rewrite $y^2 = 2x + 6$ as $x = \frac{1}{2}(y^2 - 6)$

2. Swap x and y.

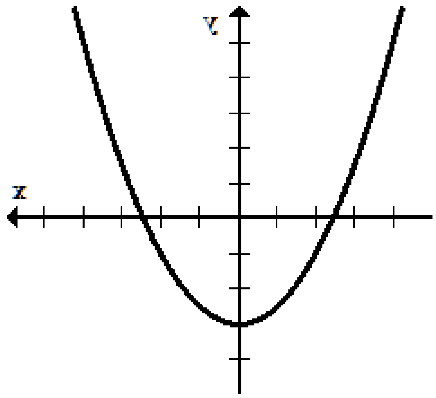
$$y = \frac{1}{2}(x^2 - 6)$$

   It's now a function of x and can graph it. You can graph this by hand or use a graphing calculator. Either way, you should get this



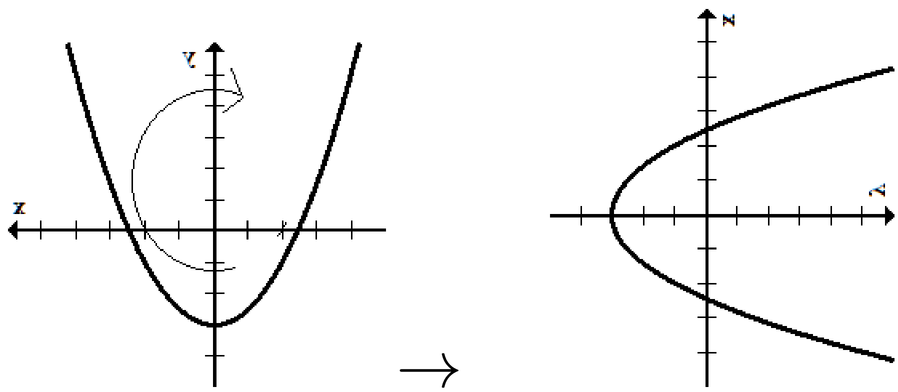
3. You need to rotate your graph along the line $y = x$. It's hard to demonstrate that on paper (much better in person), so instead here's another way.

(a) Flip your graph over the y-axis. You're graph should now look like this,

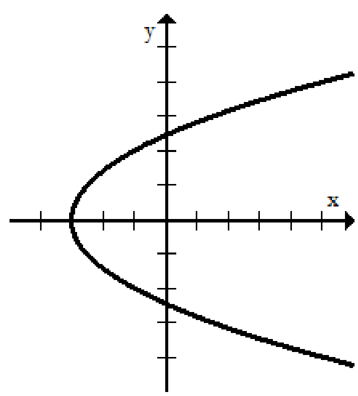


You probably can't tell a difference since the function is symmetric about the y-axis. But you can see the x-axis is now pointing to the left.

(b) Rotate your graph 90° (90 degrees) to the right.

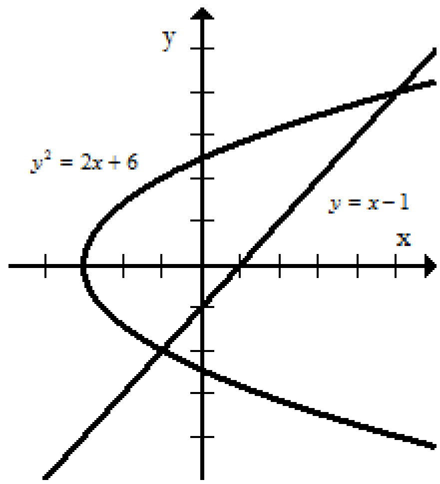


(c) You're done graphing $y^2 = 2x + 6$. Now just re-label the axes.

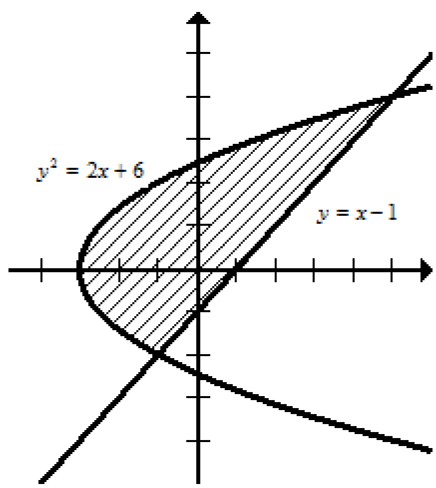


4. Since $y = x - 1$ isn't a function of y, I didn't use the method described above. If it's a function of x, wait until you're done graphing all functions of y, then just graph

$y = x - 1$ normally.



5. We shade the area enclosed by the two functions.



6. Since we are integrating with respect to y, we need to find our endpoints. At what y-value does the shaded region begin and at what y-value does the shaded region end?

$$y = -2 \text{ and } y = 4$$

How do you find this algebraically? You're looking for where the functions intersect. So set them equal to each other. BUT! you need to make sure both functions are in terms of the same variable. Rewrite them as

$$x = y + 1 \text{ and } x = \frac{1}{2}(y^2 - 6)$$

and solve

$$y + 1 = \frac{1}{2}(y^2 - 6)$$

I'll leave that to you. You should still get $y = -2$ and $y = 4$.

7. Now set up the integral and evaluate. At this point all functions have to be in terms of the same variable (in our case, y).

$$\begin{aligned} A &= \int_{-2}^{4} \left(\overset{right}{function}\right) - \left(\overset{left}{function}\right)\, dy \\ &= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right)\, dy \\ &= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\, dy \\ &= -\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\Big|_{-2}^{4} \\ &= 18 \end{aligned}$$

Wasn't that fun?!? Let's do another!