Math 230

Calculus II

Brian Veitch • Sarah Wesley • Northern Illinois University

Review

IMPORTANT: If you see any errors/typos please email asap bveitch@niu.edu.

Final Exam Practice Problem Set:

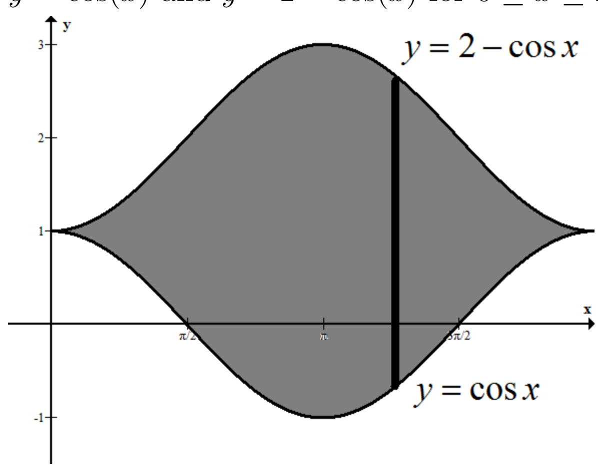
http://brianveitch.com/calculus2/finalexam/final-exam-review-wesley.pdf

Check here to see if there is a video solution to a question:

https://www.youtube.com/playlist?list=PL9kH3mCPsggdIPuugOozgnZESzEZJBs-0

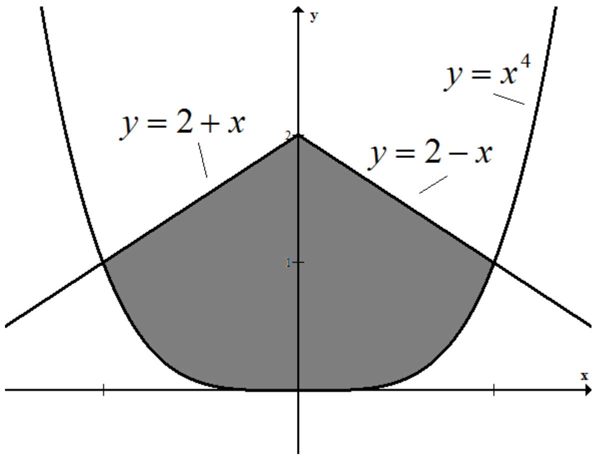
1. Sketch the region enclosed by the given curves and find its area.

   (a) $y = \cos(x)$ and $y = 2 - \cos(x)$ for $0 \le x \le 2\pi$.



$$\begin{aligned}
\int_a^b Top - Bottom\ dx &= \int_0^{2\pi} (2 - \cos x) - (\cos x)\ dx \\
&= \int_0^{2\pi} 2 - 2\cos x\ dx \\
&= 2x - 2\sin x\Big|_0^{2\pi} \\
&= [4\pi - 2\sin(2\pi)] - [0 - 2\sin(0)] \\
&= 4\pi
\end{aligned}$$

(b) $y = x^4$ and $y = 2 - |x|$



*Note: You can't integrate functions that have an absolute value as they are officially defined as piecewise functions.

$$2 - |x| = \begin{cases} 2 - x, & x \geq 0 \\ 2 - (-x) = 2 + x, & x < 0 \end{cases}$$

Since the top function changes from $y = 2 + x$ to $y = 2 - x$ at $x = 0$ we need to break this into two integrals.

$$\begin{aligned} \int_a^b Top - Bottom\ dx &= \int_{-1}^{0} (2 + x) - x^4\ dx + \int_0^{-1} (2 - x) - x^4\ dx \\ &= \left[2x + \frac{1}{2}x^2 - \frac{1}{5}x^5 \Big|_{-1}^{0} \right] + \left[2x - \frac{1}{2}x^2 - \frac{1}{5}x^5 \Big|_{0}^{1} \right] \\ &= 1.3 + 1.3 \\ &= 2.6 \end{aligned}$$

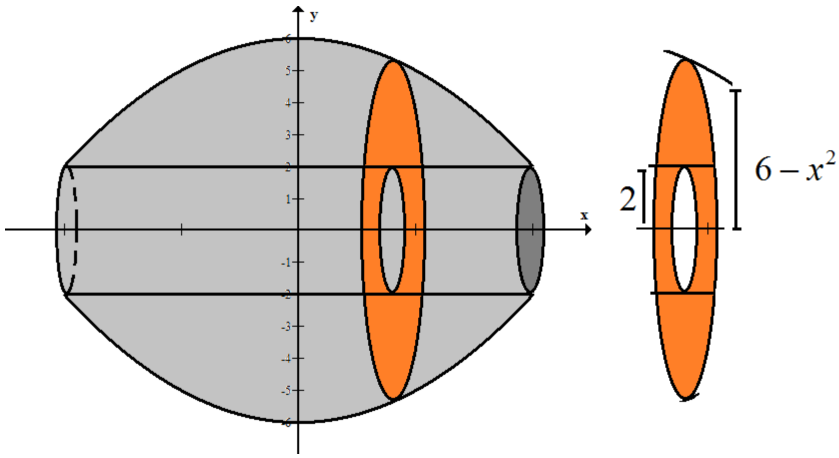
If you see that the area left of $x = 0$ is identical to the area right of $x = 0$, then you can evaluate

$$2 \int_0^1 (2 - x) - x^4\ dx$$

to find the total area.

2. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical washer/disk.

(a) $y = 6 - x^2$, $y = 2$; about the x-axis.



i. $R(x) = 6 - x^2$

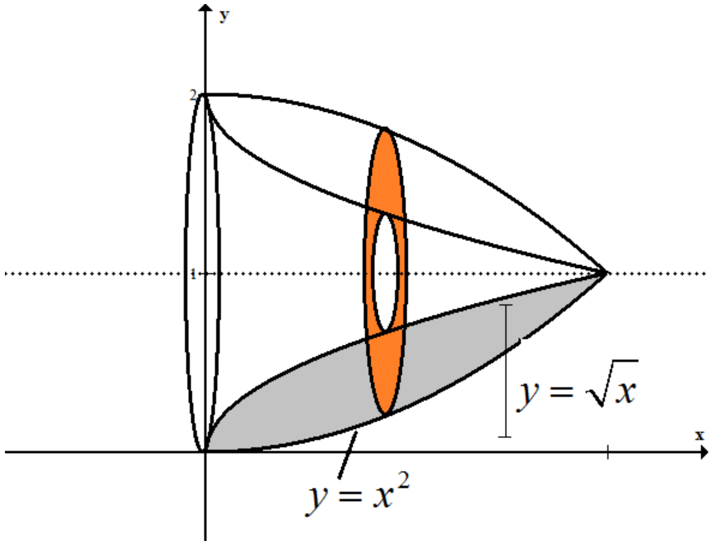
ii. $r(x) = 2$

iii. $a = -2$, $b = 2$

iv. Formula (Washer Method): $V = \int_a^b \pi(R(x))^2 - \pi(r(x))^2 \, dx$

$$\begin{aligned}
\int_a^b \pi R^2 - \pi r^2 \, dx &= \int_{-2}^{2} \pi(6 - x^2)^2 - \pi(2)^2 \, dx \\
&= \pi \int_{-2}^{2} (36 - 12x^2 + x^4) - 4 \, dx \\
&= \pi \int_{-2}^{2} 32 - 12x^2 + x^4 \, dx \\
&= \pi \left[32x - 4x^3 + \frac{1}{5}x^5 \Big|_{-2}^{2} \right] \\
&= \frac{348\pi}{5}
\end{aligned}$$

(b) $y = x^2$, $x = y^2$; about $y = 1$



*Note: The radius is the distance from the axis of revolution to the functions **through the bounded region**.

i. $R(x) = 1 - x^2$

ii. $r(x) = 1 - \sqrt{x}$

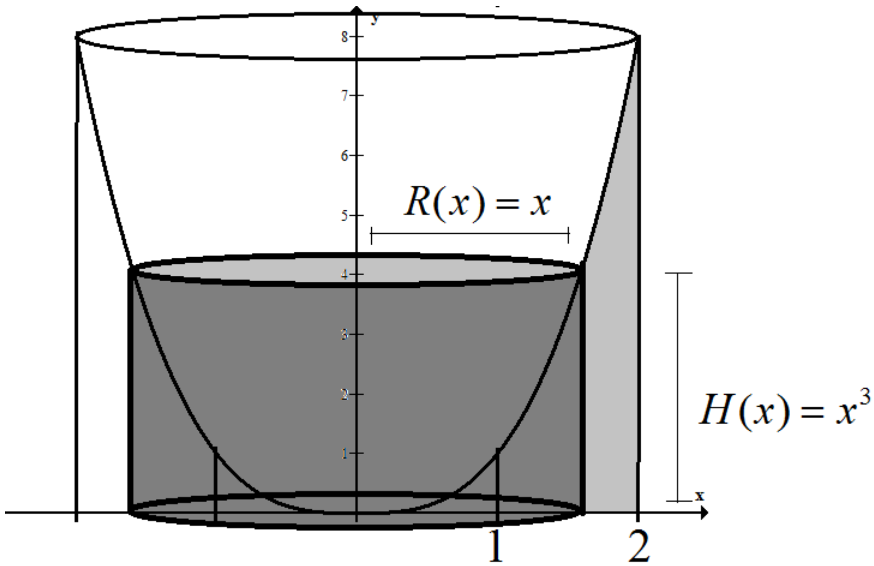
iii. $a = 0$, $b = 1$

iv. Formula (Washer Method): $V = \displaystyle\int_a^b \pi(R(x))^2 - \pi(r(x))^2 \, dx$

$$\begin{aligned}
\int_a^b \pi R^2 - \pi r^2 \, dx &= \int_0^1 \pi(1 - x^2)^2 - \pi(1 - \sqrt{x})^2 \, dx \\
&= \pi \int_0^1 (1 - 2x^2 + x^4) - (1 - 2\sqrt{x} + x) \, dx \\
&= \pi \int_0^1 x^4 - 2x^2 - x + 2x^{1/2} \, dx \\
&= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + \frac{4}{3}x^{3/2} \Big|_0^1 \right] \\
&= \frac{11\pi}{30}
\end{aligned}$$

3. Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the curves around the given line.

(a) $y = x^3$, $y = 0$, $x = 1$, and $x = 2$; around the y-axis.



i. Formula (Shell Method in terms of x): $V = \int_a^b 2\pi R(x)H(x)\,dx$

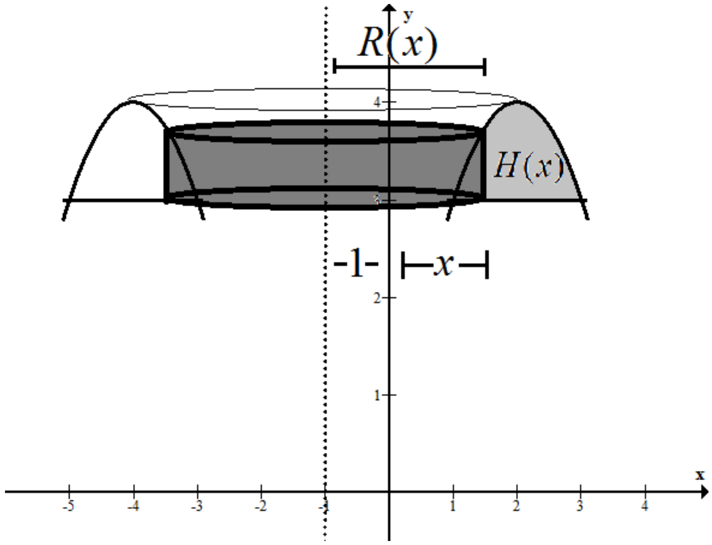
ii. $R(x) = x$

iii. $H(x) = x^3$

iv. $a = 1$, $b = 2$ (Must be endpoints from original bounded region)

$$\begin{aligned}\int_a^b 2\pi R(x)H(x)\,dx &= \int_1^2 2\pi(x)(x^3)\,dx \\ &= \int_1^2 2\pi x^4\,dx \\ &= \frac{2\pi}{5}x^5\Big|_1^2 \\ &= \frac{64\pi}{5} - \frac{2\pi}{5} \\ &= \frac{62\pi}{5}\end{aligned}$$

(b) $y = 4x - x^2$, $y = 3$; about the line $x = -1$



i. Formula (Shell Method in terms of x): $V = \int_a^b 2\pi R(x)H(x)\,dx$

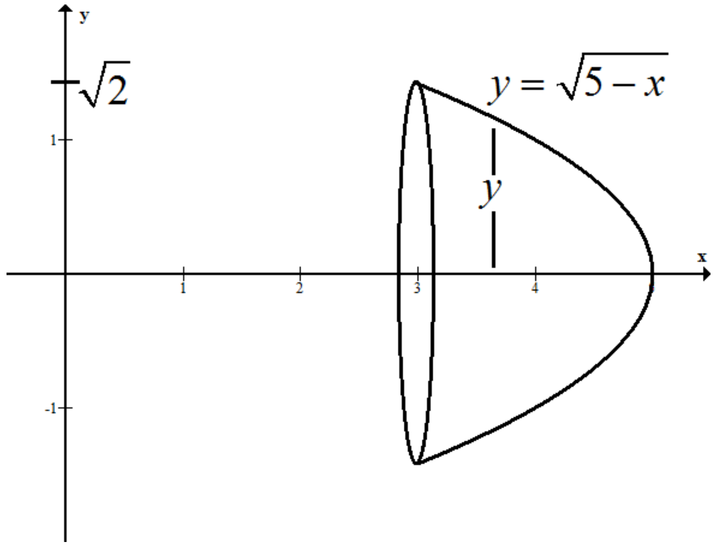
ii. $R(x) = 1 + x$

iii. $H(x) = (4x - x^2) - 3 = -x^2 + 4x - 3$

iv. $a = 1$, $b = 3$ (Must be endpoints from original bounded region)

$$\begin{aligned}
\int_a^b 2\pi R(x)H(x)\,dx &= \int_1^3 2\pi(1+x)(-x^2+4x-3)\,dx \\
&= 2\pi \int_1^3 -x^3 + 3x^2 + x - 3\,dx \\
&= 2\pi \left[-\frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 - 3x \Big|_1^3 \right] \\
&= 2\pi \left[\frac{9}{4} - \left(-\frac{7}{4} \right) \right] \\
&= 8\pi
\end{aligned}$$

4. Find the exact area of the surface obtained by rotating the curve $y = \sqrt{5-x}$ for $3 \le x \le 5$ about the x-axis.



*Note: There are two versions of the formula to find the surface area of a region rotated around the x-axis. They are

$$SA = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$SA = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The only difference is the $\frac{dy}{dy}$ or $\frac{dx}{dy}$ in the integrand. It's up to you. Normally one is easier than the other. I'll do both but I'll do the easier one first.

Write everything so it's in terms of y.

(a) Instead of $y = \sqrt{5-x}$, use $x = 5 - y^2$.

(b) $\dfrac{dx}{dy} = -2y$

$$\begin{aligned}
\int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy &= \int_0^{\sqrt{2}} 2\pi y\sqrt{1+(-2y)^2}\,dy \\
&= \int_0^{\sqrt{2}} 2\pi y\sqrt{1+4y^2}
\end{aligned}$$

$$\text{Let } u = 1+4y^2,\;\; du = 8y\,dy \;\;\rightarrow\;\; \frac{1}{8}\,du = y\,dy$$

$$\begin{aligned}
&= \int_1^9 \frac{2\pi}{8}\sqrt{u}\,du \\
&= \int_1^9 \frac{\pi}{4}u^{1/2}\,du \\
&= \frac{1}{6}u^{3/2}\Big|_1^9 \\
&= \frac{\pi}{6}(9)^{3/2} - \frac{\pi}{6}(1)^{3/2} \\
&= \frac{27\pi}{6} - \frac{\pi}{6} \\
&= \frac{26\pi}{6} \\
&= \frac{13\pi}{3}
\end{aligned}$$

Now in terms of x. Warning: some tricky simplifying is involved.

$$\begin{aligned}
\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dy &= \int_3^5 2\pi\sqrt{5-x}\sqrt{1+\left(\frac{-1}{2\sqrt{5-x}}\right)^2}\,dx \\
&= \int_3^5 2\pi\sqrt{5-x}\sqrt{1+\frac{1}{4(5-x)}}\,dx
\end{aligned}$$

Multiply the two square roots together

$$\begin{aligned}
&= \int_3^5 2\pi\sqrt{(5-x)+\frac{1}{4}}\,dx \\
&= \int_3^5 2\pi\sqrt{-x+\frac{21}{4}}\,dx
\end{aligned}$$

Let $u = -x + \frac{21}{4}$, $du = -dx \quad \rightarrow \quad -du = dx$.

$$\text{If } x = 5,\ u = \frac{1}{4}$$

$$\text{If } x = 3,\ u = \frac{9}{4}$$

$$\begin{aligned}
\int_3^5 2\pi\sqrt{-x + \frac{21}{4}}\,dx &= \int_{9/4}^{1/4} -2\pi\sqrt{u}\,du \\
&= \int_{9/4}^{1/4} -2\pi u^{1/2}\,du \\
&= -\frac{4\pi}{3}u^{3/2}\Big|_{9/4}^{1/4} \\
&= \left[-\frac{4\pi}{3}(1/4)^{3/2}\right] - \left[-\frac{4\pi}{3}(9/4)^{9/1}\right] \\
&= \frac{13\pi}{3}
\end{aligned}$$

5. Find $(f^{-1})'(a)$ given

(a) $f(x) = 3x^3 + 4x^2 + 6x + 5$, $a = 5$

$$\left(f^{-1}\right)'(5) = \frac{1}{f'(f^{-1}(5))}$$

i. To find $f^{-1}(5) = x$, we need to solve $f(x) = 5$. Usually you have to guess. Try easy values like -1, 1, 2, 0, etc.

$$3x^3 + 4x^2 + 6x + 5 = 5$$

I'd guess $x = 0$ (which works). Since we assume f has an inverse this must be only x value that will work.

$$f^{-1}(5) = 0$$

ii. Next, we need to find $f'(x)$

$$f'(x) = 9x^2 + 8x + 6$$

iii. Now we need to evaluate $f'(f^{-1}(5))$

$$f'(f^{-1}(5)) = f'(0) = 6$$

iv. Put everything together

$$\left(f^{-1}\right)'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(0)} = \frac{1}{6}$$

(b) $f(x) = x^3 + 3\sin(x) + 2\cos(x)$, $a = 2$

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(f^{-1}(2))}$$

i. To find $f^{-1}(2) = x$, we need to solve $f(x) = 2$. Usually you have to guess. Try easy values like -1, 1, 2, 0, etc.

$$x^3 + 3\sin(x) + 2\cos(x) = 2$$

I'd guess $x = 0$ (which works).

$$0^3 + 3\sin(0) + 2\cos(0) = 2$$

Since we assume f has an inverse this must be only x value that will work.

$$f^{-1}(2) = 0$$

ii. Next, we need to find $f'(x)$

$$f'(x) = 3x^2 + 3\cos(x) - 2\sin(x)$$

iii. Now we need to evaluate $f'(f^{-1}(2))$

$$f'(f^{-1}(2)) = f'(0) = 3$$

iv. Put everything together

$$\left(f^{-1}\right)'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(0)} = \frac{1}{3}$$

6. Differentiate the following functions

(a) $f(x) = x^2 e^{-1/x}$

$$\begin{aligned} f'(x) &= x^2 \cdot \left[e^{-1/x}\right]' + \left[x^2\right]' \cdot e^{-1/x} \\ &= x^2 \cdot e^{-1/x} \cdot \frac{1}{x^2} + 2x \cdot e^{-1/x} \\ &= e^{-1/x} + 2xe^{-1/x} \\ &= e^{-1/x}\,(1 + 2x) \end{aligned}$$

(b) $f(x) = \ln(\sin^2(x))$

$$\begin{aligned} f'(x) &= \frac{1}{\sin^2(x)} \cdot \left[\sin^2(x)\right]' \\ &= \frac{1}{\sin^2(x)} \cdot 2\sin(x) \cdot \cos(x) \\ &= \frac{2\cos(x)}{\sin(x)} \\ &= 2\cot(x) \end{aligned}$$

(c) $f(x) = x\sin(2^x)$

$$\begin{aligned} f'(x) &= x \cdot [\sin(2^x)]' + [x]' \cdot \sin(2^x) \\ &= x \cdot \cos(2^x) \cdot 2^x \ln(2) + 1 \cdot \sin(2^x) \\ &= x2^x \cos(2^x)\ln(2) + \sin(2^x) \end{aligned}$$

(d) $f(x) = x^x$. There are two ways of doing this. I'll do both.

i. Rewrite $f(x) = x^x$ as $f(x) = e^{x\ln x}$

$$\begin{aligned} f'(x) &= e^{x\ln x} \cdot [x\ln x]' \\ &= e^{x\ln x}\left[x \cdot \frac{1}{x} + 1 \cdot \ln x\right] \\ &= x^x\,(1 + \ln x) \end{aligned}$$

ii. Now using logarithmic differetiation with $y = x^x$

$$\begin{aligned}
\ln y &= \ln(x^x) \\
\ln y &= x\ln(x) \\
&\quad \text{Differentiate both sides} \\
\frac{1}{y}\cdot y' &= x\cdot\frac{1}{x} + 1\cdot\ln x \\
&\quad \text{Solve for } y' \\
y' &= y(1+\ln x) \\
y' &= x^x(1+\ln x)
\end{aligned}$$

(e) $f(x) = (\tan^{-1} x)^2$

$$\begin{aligned}
f'(x) &= 2\left(\tan^{-1} x\right)^1 \cdot \left[\tan^{-1} x\right]' \\
&= 2\tan^{-1} x \cdot \frac{1}{1+x^2} \\
&= \frac{2\tan^{-1} x}{1+x^2}
\end{aligned}$$

(f) $f(x) = \cos^{-1}(\sin^{-1} x))$

$$\begin{aligned}
f'(x) &= \frac{-1}{\sqrt{1-(\sin^{-1} x)^2}} \cdot \left[\sin^{-1} x\right]' \\
&= \frac{-1}{\sqrt{1-(\sin^{-1} x)^2}} \cdot \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

7. Evaluate the integral.

(a) $\displaystyle\int \frac{dx}{x\ln x}$

i. Let $u = \ln x$

ii. $du = \dfrac{1}{x}\,dx$

iii. Substitute

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|\ln x| + C\end{aligned}$$

(b) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

i. Let $u = \ln x$

ii. $du = \dfrac{1}{x}\,dx$

iii. Substitute

$$\begin{aligned}\int \frac{\cos(\ln x)}{x} &= \int \cos(u)\,du \\ &= \sin(u) + C \\ &= \sin(\ln x) + C\end{aligned}$$

(c) $\displaystyle\int x2^{x^2}\,dx$

i. Let $u = x^2$

ii. $du = 2x\,dx \quad\rightarrow\quad \frac{1}{2}\,du = x\,dx$

iii. Substitute

$$\begin{aligned}\int x2^{x^2}\,dx &= \int \frac{1}{2}2^u\,du \\ &= \frac{1}{2}\int 2^u\,du \\ &= \frac{1}{2}\cdot\frac{2^u}{\ln 2} + C \\ &= \frac{2^{x^2}}{2\ln 2} + C\end{aligned}$$

(d) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

i. Let $u = \sin^{-1} x$

ii. $du = \frac{1}{\sqrt{1-x^2}}\,dx$

iii. If $x = 0$, $u = \sin^{-1}(0) = 0$

iv. If $x = 1/2$, $u = \sin^{1/2} = \pi/6$

v. Substitute

$$\begin{aligned}
\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx &= \int_0^{\pi/6} u\,du \\
&= \frac{1}{2}u^2\Big|_0^{\pi/6} \\
&= \frac{(\pi/6)^2}{2} - \frac{0^2}{2} \\
&= \frac{\pi^2}{72}
\end{aligned}$$

8. Evaluate the limits.

(a) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \frac{e^0 - 1 - 0}{0^2} = \frac{0}{0} \text{ Indeterminate} \\
&\overset{LH}{=} \lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{0}{0} \text{ Indeterminate} \\
&\overset{LH}{=} \lim_{x\to 0} \frac{e^x}{2} \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\lim\limits_{x\to 1^+} x^{1/(1-x)}$

$$\lim_{x\to 1^+} x^{1/(1-x)} = 1^\infty \text{ Indeterminate}$$

Rewrite $x^{1/(1-x)}$ as $e^{\frac{1}{1-x}\ln x}$ using the rule $a^u = e^{u \ln a}$. Now focus on evaluating $\lim\limits_{x\to 1^+} \dfrac{1}{1-x} \ln x$

$$\begin{aligned}
\lim_{x\to 1^+} \frac{1}{1-x}\ln x &= \lim_{x\to 1^+} \frac{\ln x}{1-x} = \frac{0}{0} \text{ Indeterminate} \\
&\overset{LH}{=} \lim_{x\to 1^+} \frac{1/x}{-1} \\
&= -1 \text{ (not final answer)}
\end{aligned}$$

This was just the value of the exponent. The final answer is

$$\lim_{x\to 1^+} x^{1/(1-x)} = e^{-1}$$

(c) $\displaystyle\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(\sqrt{x})}{x^2} &= \frac{\infty}{\infty} \text{ (Indeterminate)} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{2x} \\
&= \lim_{x\to\infty} \frac{\frac{1}{2x}}{2x} \\
&= \lim_{x\to\infty} \frac{1}{4x^2} \\
&= 0
\end{aligned}$$

9. Evaluate the integral.

(a) $\displaystyle\int xe^{-3x}\,dx$ (Integration By Parts)

$$u = x, \qquad dv = e^{-3x}\,dx$$

$$du = dx, \quad v = -\frac{1}{3}e^{-3x}$$

Formula: $\displaystyle\int u\,dv = uv - \int v\,du$

$$\begin{aligned}
\int xe^{-3x}\,dx &= x\left(-\frac{1}{3}e^{-3x}\right) - \int\left(-\frac{1}{3}e^{-3x}\right)\,dx \\
&\quad \text{Clean it up} \\
&= -\frac{1}{3}xe^{-3x} + \int\frac{1}{3}e^{-3x}\,dx \\
&= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C
\end{aligned}$$

(b) $\displaystyle\int (x^2+2x)\cos(x)\,dx$

$$u = x^2 + 2x \qquad dv = \cos(x)\ dx$$

$$du = (2x + 2)\ dx \qquad v = \sin(x)$$

Formula: $\displaystyle\int u\ dv = uv - \int v\ du$

$$\begin{aligned}
\int (x^2 + 2x)\cos(x)\ dx &= (x^2 + 2x)\sin(x) - \int (2x + 2)\sin(x)\ dx \\
&= (x^2 + 2x)\sin(x) - \left[\overset{u=2x+2}{du = 2\ dx} \quad \overset{dv=\sin(x)}{v = -\cos(x)} \right] \\
&= (x^2 + 2x)\sin(x) - \left[-(2x + 2)\cos(x) - \int -2\cos(x)\ dx \right] \\
&= (x^2 + 2x)\sin(x) + (2x + 2)\cos(x) + \int -2\cos(x)\ dx \\
&= (x^2 + 2x)\sin(x) + (2x + 2)\cos(x) - 2\sin(x) + C
\end{aligned}$$

(c) $\displaystyle\int_0^{\pi/2} \sin^7(\theta)\cos^5(\theta)\ d\theta$

Since the power of $\cos(\theta)$ is odd, factor out one $\cos(\theta)$

$$\begin{aligned}
\int_0^{\pi/2} \sin^7(\theta)\cos^5(\theta)\ d\theta &= \int_0^{\pi/2} \sin^7(\theta)\cos^4(\theta) \cdot \cos(\theta)\ d\theta \\
&= \int_0^{\pi/2} \sin^7(\theta)\left(1 - \sin^2(\theta)\right)^2 \cdot \cos(\theta)\ d\theta
\end{aligned}$$

Let $u = \sin(\theta), \;\; du = \cos(\theta)$

If $\theta = \pi/2, \;\; u = \sin(\pi/2) = 1$

If $\theta = 0, \;\; u = \sin(0) = 0$

$$\begin{aligned}
&= \int_0^1 u^7(1-u^2)^2 \, du \\
&= \int_0^1 u^7(1-2u^2+u^4) \, du \\
&= \int_0^1 u^7 - 2u^9 + u^{11} \, du \\
&= \frac{1}{8}u^8 - \frac{1}{5}u^{10} + \frac{1}{12}u^{12}\Big|_0^1 \\
&= \left[\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right] - [0 - 0 + 0] \\
&= \frac{1}{120}
\end{aligned}$$

(d) $\int \tan(x)\sec^3(x) \, dx$

Since the power of $\tan(x)$ is odd, factor out $\sec(x)\tan(x)$

$$\int \tan(x)\sec^3(x) \, dx = \int \sec^2(x) \cdot \sec(x)\tan(x) \, dx$$

Let $u = \sec(x), \;\; du = \sec(x)\tan(x) \, dx$

$$\begin{aligned}
&= \int u^2 \, du \\
&= \frac{1}{3}u^3 + C \\
&= \frac{1}{3}\sec^3(x) + C
\end{aligned}$$

(e) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ (Trig Substitution)

$$\text{Let } x = 3\sin(\theta), \quad dx = 3\cos(\theta)\,d\theta$$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{9\sin^2(\theta)}{\sqrt{9-9\sin^2(\theta)}}3\cos(\theta)\,d\theta \\
&= \int \frac{27\sin^2(\theta)\cos(\theta)}{\sqrt{9\cos^2(\theta)}}\,d\theta \\
&= \int \frac{27\sin^2(\theta)\cos(\theta)}{3\cos(\theta)}\,d\theta \\
&= \int 9\sin^2(\theta)\,d\theta
\end{aligned}$$

Trig Identity Needed: $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$

$$\begin{aligned}
\int 9\sin^2(\theta)\,d\theta &= \int \frac{9}{2}(1-\cos(2\theta))\,d\theta \\
&= \int \frac{9}{2} - \frac{9}{2}\cos(2\theta)\,d\theta \\
&= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C
\end{aligned}$$

Trig Identity: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

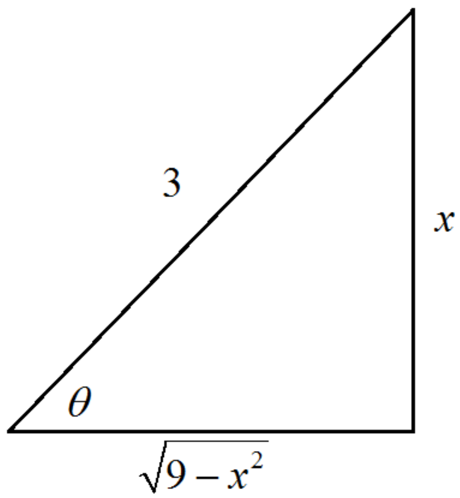
$$= \frac{9}{2}\theta - \frac{9}{2}\sin(\theta)\cos(\theta) + C$$

Need to get back to a function of x.

$$\text{If } x = 3\sin(\theta), \quad \text{then } \frac{x}{3} = \sin(\theta)$$

Solve for θ and we get $\theta = \sin^{-1}\left(\frac{x}{3}\right)$. This takes care of the first part of the antiderviative $\dfrac{9}{2}\theta = \dfrac{9}{2}\sin^{-1}\left(\dfrac{x}{3}\right)$

To get $\sin(\theta)\cos(\theta)$ back in terms of x, we use the following triangle.



From the triangle: $\sin(\theta) = \dfrac{x}{3}$ and $\cos(\theta) = \dfrac{\sqrt{9-x^2}}{3}$

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C \\ &= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C\end{aligned}$$

(f) $\displaystyle\int \frac{\sqrt{x^2-1}}{x^4}\,dx$ (Trig Substitution)

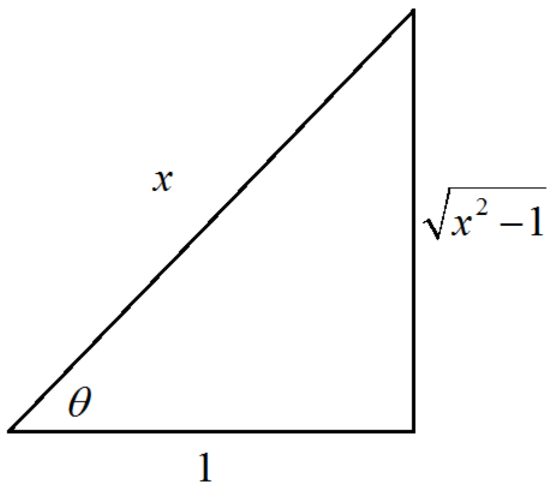
$$\text{Let } x = \sec(\theta), \quad dx = \sec(\theta)\tan(\theta)\,d\theta$$

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x^4}\,dx &= \int \frac{\sqrt{\sec^2(\theta)-1}}{\sec^4(\theta)}\cdot\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{\sqrt{\tan^2(\theta)}}{\sec^4(\theta)}\cdot\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{\tan(\theta)}{\sec^4(\theta)}\cdot\sec(\theta)\tan(\theta)\,d\theta \\ &= \int \frac{\tan^2(\theta)}{\sec^3(\theta)}\,d\theta \\ &= \int \sin^2(\theta)\cos(\theta)\end{aligned}$$

$$\text{Let } u = \sin(\theta), \quad du = \cos(\theta)\,d\theta$$

$$\begin{aligned}\int \sin^2(\theta)\cos(\theta)\,d\theta &= \int u^2\,du\\ &= \frac{1}{3}u^3 + C\\ &= \frac{1}{3}\sin^3(\theta) + C\end{aligned}$$

To get $\sin^3(\theta)$ back in terms of x, we use the following triangle. Note from the original substitution $\frac{x}{1} = \sec(\theta)$



From the triangle: $\sin(\theta) = \dfrac{\sqrt{x^2-1}}{x}$

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C = \frac{(x^2-1)^{3/2}}{3x^3} + C$$

(g) $\int_0^{1/2} x\sqrt{1-4x^2}\,dx$ (u-substitution)

i. Let $u = 1 - 4x^2$

ii. $du = -8x\,dx \quad \to \quad -\dfrac{1}{8}\,du = x\,dx$

iii. If $x = 0$, $u = 1 - 4(0)^2 = 1$

iv. If $x = 1/2$, $u = 1 - 4(1/2)^2 = 0$

v. Substitute

$$\begin{aligned}
\int_0^{1/2} x\sqrt{1-4x^2}\,dx &= \int_1^0 -\frac{1}{8}\sqrt{u}\,du \\
&= \int_1^0 -\frac{1}{8}u^{1/2}\,du \\
&= -\frac{1}{12}u^{3/2}\Big|_1^0 \\
&= \left[-\frac{1}{12}(0)^{3/2}\right] - \left[-\frac{1}{12}(1)^{3/2}\right] \\
&= \frac{1}{12}
\end{aligned}$$

(h) $\int \dfrac{5x+1}{(2x+1)(x-1)}\,dx$ (Partial Fractions)

i. Factor the denominator of $\dfrac{5x+1}{(2x+1)(x-1)}$. Note, $(2x+1)(x-1)$ is already factored.

ii. We want to write $\dfrac{5x+1}{(2x+1)(x-1)}$ as $\dfrac{A}{2x+1} + \dfrac{B}{x-1}$

iii. Multiply both sides by $(2x+1)(x-1)$ to clear the denominators.

$$5x + 1 = A(x-1) + B(2x+1)$$

iv. Shortcut: Plug in $x = 1$ to solve for B.

$$6 = B(3) \quad \to \quad B = 2$$

v. Shortcut: Plug in $x = -1/2$ to solve for A

$$-\frac{3}{2} = A\left(-\frac{3}{2}\right) \quad \to \quad A = 1$$

vi. Rewrite the integral and evaluate

$$\begin{aligned}\int \frac{5x+1}{(2x+1)(x-1)}\,dx &= \int \frac{1}{2x+1} + \frac{2}{x-1}\,dx \\ &= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C\end{aligned}$$

(i) $\displaystyle\int \frac{4x}{x^3 - x^2 - x + 1}\,dx$

i. Factor the denominator of $\dfrac{4x}{x^3 - x^2 - x + 1}$.
Note, $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$

ii. We want to write $\dfrac{4x}{(x-1)^2(x+1)}$ as $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1}$

iii. Multiply both sides by $(x-1)^2(x+1)$ to clear the denominators.

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

iv. Shortcut: Plug in $x = 1$ to solve for B.

$$4 = B(2) \quad \rightarrow \quad B = 2$$

v. Shortcut: Plug in $x = -1$ to solve for C

$$-4 = C(-2)^2 \quad \rightarrow \quad C = -1$$

vi. We still need A. Multiply out the right hand side and you'll get

$$4x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

Match the coefficients on each side. Since the left hand side has no x^2 term, we know

$$0x^2 = (A+C)x^2$$

$$0 = (A-1)$$

$$1 = A$$

vii. Rewrite the integral and evaluate

$$\begin{aligned}\int \frac{4x}{(x-1)^2(x+1)}\,dx &= \int \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\,dx \\ &= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C\end{aligned}$$

10. Evaluate the improper integrals.

(a) $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx$

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx$$

i. Let $u = 1 + x$

ii. $du = dx$

iii. If $x = t$, then $u = 1 + t$

iv. If $x = 0$, then $u = 1 + 0 = 1$

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t \frac{1}{\sqrt[4]{1+x}}\,dx &= \lim_{t\to\infty}\int_1^{1+t}\frac{1}{\sqrt[4]{u}}\,du \\
&= \lim_{t\to\infty}\int_1^{1+t} u^{-1/4}\,du \\
&= \lim_{t\to\infty}\left[\frac{4}{3}u^{3/4}\Big|_1^{1+t}\right] \\
&= \lim_{t\to\infty}\left[\frac{4}{3}(1+t)^{3/4} - \frac{4}{3}(1)^{3/4}\right] \\
&= \infty
\end{aligned}$$

Diverges

(b) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx$$

Since I have to use integration by parts, I'm going to drop the limit notation for now.

$$u = \ln x, \quad dv = \frac{1}{x^2}\,dx$$
$$du = \frac{1}{x}\,dx, \quad v = -\frac{1}{x}$$

$$\begin{aligned}
\int \frac{\ln x}{x^2}\,dx &= -\frac{1}{x}\ln x - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx \\
&\quad \text{clean it up} \\
&= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx \\
&= -\frac{\ln x}{x} - \frac{1}{x} \\
&= \frac{-\ln x - 1}{x}
\end{aligned}$$

Let's add back in the bounds and the limit.

$$\begin{aligned}
\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx &= \lim_{t\to\infty}\left[\left.\frac{-\ln x - 1}{x}\right|_1^t\right] \\
&= \lim_{t\to\infty}\left[\frac{-\ln t - 1}{t} - \frac{-\ln 1 - 1}{1}\right] \\
&= \lim_{t\to\infty}\left[\frac{-\ln t - 1}{t}\right] + 1 \\
&= 0 + 1 \\
&= 1
\end{aligned}$$

Note that $\lim_{t\to\infty}\frac{-\ln t - 1}{t} = \frac{-\infty}{\infty}$ is indeterminate. Use LH rule.

$$\lim_{t\to\infty}\frac{-\ln t - 1}{t} \overset{LH}{=} \lim_{t\to\infty}\frac{-1/t}{1} = 0$$

11. Let $a_n = \frac{2n}{3n+1}$.

(a) Determine whether a_n is convergent.

a_n is a sequence and will converge if $\lim_{n\to\infty} a_n = L$ where L is a finite number.

$$\lim_{n\to\infty}\frac{2n}{3n+1} \overset{LH}{=} \lim_{n\to\infty}\frac{2}{3} = \frac{2}{3}$$

. Yes, a_n is convergent. It converges to $\frac{2}{3}$.

(b) Determine whether $\sum a_n$ is convergent.

In order for any series $\sum a_n$ to converge the following requirement MUST be met.

$$\lim_{n\to\infty} a_n = 0$$

If this requirement isn't met then $\sum a_n$ is divergent. This is called **The Divergence Test**. Since $\displaystyle\lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \neq 0$, the series $\displaystyle\sum \frac{2n}{3n+1}$ MUST diverge.

IMPORTANT!! $\displaystyle\lim_{n\to\infty} a_n = 0$ DOES NOT mean $\sum a_n$ converges. If this was true, we wouldn't need all those series convergence tests we spent so much time learning.

(c) What requirement do you need on the sequence a_n for $\sum a_n$ to converge?

In order for any series $\sum a_n$ to converge the following requirement MUST be met.

$$\lim_{n\to\infty} a_n = 0$$

IMPORTANT!! $\displaystyle\lim_{n\to\infty} a_n = 0$ DOES NOT mean $\sum a_n$ converges. If this was true, we wouldn't need all those series convergence tests we spent so much time learning.

12. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n+1}}$ (uses AST and LCT)

Let $b_n = \dfrac{1}{\sqrt{n+1}}$

i. $b_n > 0$. Check.

ii. b_n is decreasing. Check.

$$(b_n)' = -\frac{1}{2(n+1)^{3/2}} < 0 \text{ for } n \geq 1$$

iii. Does $b_n \to 0$? Check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

This means $\sum \frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges by the Alternating Series Test. But we need to check if it's absolute or conditional.

Definition: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

i. Compare our series to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (which diverges by p-test).

ii. We can't use the Direct Comparison Test because $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$.

iii. Use the Limit Comparison Test. Let $b_n = \frac{1}{\sqrt{n+1}}$ and $c_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 1$$

The Limit Comparison Test states that if $\lim_{n\to\infty} \frac{b_n}{c_n} = L > 0$ then both $\sum b_n$ and $\sum c_n$ converge or they both diverge. Since $\sum \frac{1}{\sqrt{n}}$ diverges by p-test, then $\sum \frac{1}{\sqrt{n+1}}$ diverges.

Final Answer: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges conditionally.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

Since $\lim_{n\to\infty} \frac{n^2}{n^2+n+1} = 1 \neq 0$, the series diverges by the Divergence Test.

(c) $\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$. (Ratio Test)

Let $a_n = \frac{n^{10}}{(-10)^{n+1}}$ and $a_{n+1} = \frac{(n+1)^{10}}{(-10)^{n+2}}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{(-10)^{n+2}}\cdot\frac{(-10)^{n+1}}{n^{10}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{n^{10}}\cdot\frac{(-10)^{n+1}}{(-10)^{n+2}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^{10}}{n^{10}}\cdot\frac{1}{-10}\right| \\
&= \left|1\cdot -\frac{1}{10}\right| \\
&= \frac{1}{10} \\
&< 1
\end{aligned}$$

Since $\frac{1}{10} < 1$ the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$ converges absolutely by the Ratio Test.

(d) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ (Root Test)

Root Test: If $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L<1$, then the series will converge absolutely. If $L=1$ the test is inconclusive and if $L>1$ the series diverges.

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n}=\lim_{n\to\infty}\frac{n^2+1}{2n^2+1}=\frac{1}{2}<1$$

The series converges absolutely.

(e) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ (Integral Test)

i. Evaluate $\displaystyle\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$

$$\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x\sqrt{\ln x}}dx$$

ii. Let $u=\ln x$

iii. $du=\dfrac{1}{x}\,dx$

iv. If $x=t$, $u=\ln t$

v. If $x=2$, $u=\ln 2$

$$\begin{aligned}\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx &= \lim_{t\to\infty}\int_2^{t}\frac{1}{x\sqrt{\ln x}}dx\\ &= \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{\sqrt{u}}\,du\\ &= \lim_{t\to\infty}\left[2\sqrt{u}\Big|_{\ln 2}^{\ln t}\right]\\ &= \lim_{t\to\infty}\left[\sqrt{\ln t}-\sqrt{\ln 2}\right]\\ &= \infty-\sqrt{\ln 2}\\ &= \infty\end{aligned}$$

Since the integral $\displaystyle\int_2^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx$ diverges then by the Integral Test so must the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$

(f) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+5}$ (uses AST and LCT)

Let $b_n = \dfrac{\sqrt{n}}{n+5}$

i. $b_n > 0$. Check.

ii. b_n is decreasing. Check.

$$(b_n)' = \frac{(n+5)\cdot\dfrac{1}{2\sqrt{n}} - \sqrt{n}}{(n+5)^2} = \frac{\frac{\sqrt{n}}{2} + \frac{5}{2\sqrt{n}} - \sqrt{n}}{(n+5)^2} = \frac{-\frac{\sqrt{n}}{2} + \frac{5}{2\sqrt{n}}}{(n+5)^2}$$

$$= \frac{-\frac{1}{2\sqrt{n}}(n-5)}{(n+5)^2} < 0 \text{ for } n \geq 5$$

iii. Does $b_n \to 0$? Check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\sqrt{n}}{n+5} = 0$$

This means $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+5}$ converges by the Alternating Series Test. But we need to check if it's absolute or conditional.

Definition: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{\sqrt{n}}{n+5}\right| = \sum_{n=1}^{\infty}\frac{\sqrt{n}}{n+5}$$

i. Compare our series to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ (which diverges by p-test).

ii. We can't use the Direct Comparison Test because $\dfrac{\sqrt{n}}{n+5} < \dfrac{1}{\sqrt{n}}$.

iii. Use the Limit Comparison Test. Let $b_n = \frac{\sqrt{n}}{n+5}$ and $c_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n+5}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{n}{n+5} = 1$$

The Limit Comparison Test states that if $\displaystyle\lim_{n\to\infty}\frac{b_n}{c_n} = L > 0$ then both $\sum b_n$ and $\sum c_n$ converge or they both diverge. Since $\sum \dfrac{1}{\sqrt{n}}$ diverges by p-test, then $\sum \dfrac{\sqrt{n}}{n+5}$ diverges.

Final Answer: $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n+5}$ converges conditionally.

13. Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$.

(a) Start by using the Ratio Test.
Let $a_n = \frac{(x-2)^n}{n^2+1}$ and $a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\cdot\frac{n^2+1}{(n+1)^2+1}\right| \\
&= \lim_{n\to\infty}\left|(x-2)\cdot\frac{n^2+1}{(n+1)^2+1}\right| \\
&= |x-2|\cdot\lim_{n\to\infty}\left|\frac{n^2+1}{(n+1)^2+1}\right| \\
&= |x-2|\cdot 1 \\
&= |x-2|
\end{aligned}$$

(b) Solve $|x-2| < 1$

$$\begin{aligned}
|x-2| &< 1 \\
-1 < x-2 &< 1 \\
1 < x &< 3
\end{aligned}$$

(c) Need to check the endpoints
$x = 3$**:** $\sum_{n=1}^{\infty}\frac{(3-2)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{1}{n^2+1} < \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by p-test and DCT.
$x = 1$**:** $\sum_{n=1}^{\infty}\frac{(1-2)^n}{n^2+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}$ converges AST.

(d) Final answer:

Center: $a = 2$

Radius of Convergence: $R = 1$

Interval of Convergence: $I = [1, 3]$

14. Find the radius of convergence and interval of convergence of the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\sqrt[3]{n}}$

(a) Start by using the Ratio Test.

Let $a_{n+1} = \dfrac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}$ and $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{\sqrt[3]{n+1}}\cdot\frac{\sqrt[3]{n}}{(-1)^n x^n}\right| \\
&= \lim_{n\to\infty}\left|(-1)x\cdot\frac{\sqrt[3]{n}}{\sqrt[3]{n+1}}\right| \\
&= |x|\cdot 1 \\
&= |x|
\end{aligned}$$

(b) Solve $|x| < 1$

$$\begin{aligned}
|x| &< 1 \\
-1 < \ x &< 1 \\
-1 < \ x &< 1
\end{aligned}$$

(c) Need to check the endpoints

$x = -1$**:** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$ diverges by p-test.

$x = 1$**:** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n 1^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$ converges by AST.

(d) Final answer:

Center: $a = 0$

Radius of Convergence: $R = 1$

Interval of Convergence: $I = (-1, 1]$

15. Find a power series representation for $f(x) = \ln(5 - x)$ and determine the radius of convergence.

Note: We want a function that has the form $f(x) = \dfrac{a}{1-r}$ so we can use the fact that

$$\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$$

(a) Note that $\frac{d}{dx}\left[\ln(5-x)\right] = \frac{-1}{5-x} = \frac{-1}{1(1-x/5)} = \frac{-1/5}{1-x/5}$

(b) This means $a = -1/5$ and $r = x/5$.

$$\frac{d}{dx}\left[\ln(5-x)\right] = \frac{-1/5}{1-x/5} = \sum_{n=0}^{\infty} -\frac{1}{5}\left(\frac{x}{5}\right)^n = \sum_{n=0}^{\infty} -\frac{x^n}{5^{n+1}}$$

(c) To get back to $f(x) = \ln(5-x)$ we need to integrate back

$$\ln(5-x) = \int \sum_{n=0}^{\infty} -\frac{x^n}{5^{n+1}}\, dx = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{(n+1)5^{n+1}} + C$$

(d) To find C, plug in the center $x = 0$

$$\ln(5-0) = \sum_{n=0}^{\infty} -\frac{0^{n+1}}{(n+1)5^{n+1}} + C$$
$$\ln 5 = 0 + C$$
$$\ln 5 = C$$

(e) Final Answer:

$$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}$$

or

$$\ln(5-x) = \ln 5 - \sum_{n=1}^{\infty} \frac{x^n}{n5^n}$$