This study guide is in no way exhaustive. As stated in class, any type of question from class, quizzes, exams, and homeworks are fair game.

1. **Some Algebra Review**

   (a) **Logarithm Properties**

   i. $y = \log_b x$ is equivalent to $x = b^y$.

   ii. $\log_b b = 1$

   iii. $\log_b 1 = 0$, $\ln 1 = 0$

   iv. $\log_b(b^r) = r$, $\ln(e^r) = r$

   v. $\log_b(x^r) = r\log_b(x)$, $\ln(x^r) = r\ln(x)$

   vi. $\log_b(MN) = \log_b(M) + \log_b(N)$

   vii. $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

   viii. The domain of $\log_b(u)$ is $u > 0$

   ix. All the rules above hold for $\ln x$. Remember that $\ln x = \log_e x$

   x. Change of Base
   $$\log_a(M) = \frac{\ln M}{\ln a} = \frac{\log_b M}{\log_b a}$$
   This is useful if you're asked to find the derivative of $y = \log_5(x+1)$

   (b) **Exponential Properties**

   i. $a^n a^m = a^{n+m}$

   ii. $(a^n)^m = a^{nm}$

   iii. $(ab)^n = a^n b^n$

   iv. $\frac{a^n}{a^m} = a^{n-m} = \frac{1}{a^{m-n}}$

   v. $a^{-n} = \frac{1}{a^n}$

   vi. $\frac{1}{a^{-n}} = a^n$

   vii. $a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$

   (c) **Properties of Radicals**

   i. $\sqrt[n]{a} = a^{1/n}$

   ii. $\sqrt[n]{a^m} = a^{m/n}$

2. **Limits**

   (a) **Notation**

   i. General Limit Notation: $\lim_{x\to a} f(x) = L$

   ii. Left Hand Limit: $\lim_{x\to a^-} f(x) = L$

   iii. Right Hand Limit: $\lim_{x\to a^+} f(x) = L$

   iv. If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ then $\lim_{x\to a} f(x) = L$

(b) **Limits at $\pm\infty$.** Assume all polynomials are in descending order.

i. $\lim_{x\to\infty} \frac{a}{x^r} = 0,\ r > 0$

ii. If $n > m$, then $\lim_{x\to\infty} \frac{ax^m + ...}{bx^n + ...} = 0$

iii. $\lim_{x\to\infty} \frac{ax^n + ...}{bx^n + ...} = \frac{a}{b}$

iv. If $m > n$, then $\lim_{x\to\infty} \frac{ax^m + ...}{bx^n + ...} = \pm\infty$

(c) **Evaluation Techniques**

i. If $f(x)$ is continuous, then $\lim_{x\to a} f(x) = f(a)$

$$\lim_{x\to 3} \sqrt{x^2+4} = \sqrt{3^2+4} = \sqrt{13}$$

ii. **Factor and Cancel**

If you evaluate a rational function and get $\frac{0}{0}$, then try to factor.

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3} \frac{(x+3)}{1} = 6$$

iii. **Rationalizing Numerators / Denominators**

Try this technique if you have radicals. Multiply top and bottom by the conjugate.

$$\lim_{x\to 4} \frac{\sqrt{x}-4}{x-16} = \lim_{x\to 4} \frac{\sqrt{x}-4}{x-16} \cdot \frac{\sqrt{x}+4}{\sqrt{x}+4} = \lim_{x\to 4} \frac{x-16}{(x-16)(\sqrt{x}+4)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+4}$$

iv. **Combine by Using Common Denominators**

Try this when you need to combine fractions within fractions

$$\lim_{x\to a} \frac{\frac{2}{x+4} - \frac{2}{a+4}}{x-a} = \lim_{x\to a} \frac{\frac{2}{x+4} - \frac{2}{a+4}}{x-a} \cdot \frac{(x+4)(a+4)}{(x+4)(a+4)} = \lim_{x\to a} \frac{2(a+4)-2(x+4)}{(x-a)(x+4)(a+4)}$$

$$= \lim_{x\to a} \frac{2a-2x}{(x-a)(x+4)(a+4)} = \lim_{x\to a} \frac{-2(x-a)}{(x-a)(x+4)(a+4)} = \lim_{x\to a} \frac{-2}{(x+4)(a+4)} = \frac{-2}{(a+4)(a+4)}$$

v. **L'Hospital's Rule and limits can be found later in this guide.**

(d) **Piecewise Functions -** Know how to graph and evaluate Piecewise Functions. These are good ones to test your understanding of left-hand, right-hand, and general limits. They are also used to test your understanding of continuity.

(e) **Definition of Continuity**

A function f is continuous at $x = a$ if the following three conditions are satisfied:

i. $f(a)$ must exist

ii. $\lim_{x\to a} f(x)$ must exist

iii. $\lim_{x\to a} f(x) = f(a)$

3. **Derivatives**

(a) **Limit Definition of a Derivative**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

(b) **Tangent Lines**

When asked to find the equation of a tangent line on f at $x = a$, you need two things: A point and a slope.

i. You're usually given the x value. If they don't tell you the y value, you must plug the x value into $f(x)$ to get the y value. Now you have a point $(a, f(a))$

ii. To find the slope m, you find $m = f'(a)$.

iii. The equation of the tangent line is $y - f(a) = f'(a)(x - a)$. You may need to write this in slope-intercept form.

(c) **Derivative Formulas**

i. $\frac{d}{dx}(c) = 0$

ii. $\frac{d}{dx}(f \pm g) = f' \pm g'$

iii. $\frac{d}{dx}(x) = 1$

iv. $\frac{d}{dx}(kx) = k$

v. Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

vi. $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1} \cdot f'(x)$

vii. Product Rule: $(f\ g)' = f'g + fg'$

viii. Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

ix. Chain Rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

x. $\frac{d}{dx}(a^x) = a^x \ln a$

xi. $\frac{d}{dx}(e^x) = e^x$

xii. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

xiii. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

xiv. $\frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \cdot e^{f(x)}$

xv. $\frac{d}{dx}(\ln(f(x))) = \frac{1}{f(x)} \cdot f'(x)$

xvi. $\frac{d}{dx}(\sin x) = \cos x$

xvii. $\frac{d}{dx}(\cos x) = -\sin x$

xviii. $\frac{d}{dx}(\tan x) = \sec^2 x$

xix. $\frac{d}{dx}(\sec x) = \sec x\ \tan x$

xx. $\frac{d}{dx}(\csc x) = -\csc x\ \cot x$

xxi. $\frac{d}{dx}(\cot x) = -\csc^2 x$

xxii. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

xxiii. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

xxiv. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

xxv. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$

xxvi. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$

xxvii. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$

REMEMBER: all of these derivatives have a version for the chain rule. For example:

$$\frac{d}{dx}\left(\tan^{-1}(e^x)\right) = \frac{1}{1+(e^{2x})^2} \cdot 2e^{2x}$$

(d) **Critical Points**

$x = c$ is a value of $f(x)$ if $f'(c) = 0$ or $f'(c)$ does not exist.

(e) **Increasing / Decreasing**

i. If $f'(x) > 0$ on an interval I, then $f(x)$ is increasing.

ii. If $f'(x) < 0$ on an interval I, then $f(x)$ is decreasing.

(f) **Concave Up / Concave Down**

i. If $f''(x) > 0$ on an interval I, then $f(x)$ is concave up.

ii. If $f''(x) < 0$ on an interval I, then $f(x)$ is concave down.

(g) **Inflection Points**

$x = c$ is an inflection point of f if

i. The point at $x = c$ must exist.

ii. Concavity changes at $x = c$

(h) **First Derivative Test to find Local (Relative) Extrema**

i. Find all critical value of $f(x)$ where $f'(c) = 0$

ii. Local Max at $x = c$ if $f'(x)$ changes from $(+)$ to $(-)$ at $x = c$.

iii. Local Min at $x = c$ if $f'(x)$ changes from $(-)$ to $(+)$ at $x = c$.

iv. If $f'(x)$ does not change signs, it's still an important point to plot. It may be a place where the slope is 0, a corner, an asymptote, a vertical tangent line, etc.

v. **Make sure you write your relative max and mins as points** $(c, f(c))$

(i) **Second Derivative Test to find Local (Relative) Extrema**

i. Find all critical value of $f(x)$ where $f'(c) = 0$

ii. Local Max at $x = c$ if $f''(c) < 0$

iii. Local Min at $x = c$ if $f''(x) > 0$

iv. **Make sure you write your relative max and mins as points** $(c, f(c))$

(j) **Absolute Extrema**

i. $(c, f(c))$ is an absolute maximum of $f(x)$ if $f(c) \geq f(x)$ for all x in the domain.

ii. $(c, f(c))$ is an absolute minimum of $f(x)$ if $f(c) \leq f(x)$ for all x in the domain.

(k) **Finding Absolute Extrema of a continuous** $f(x)$ **over** $[a, b]$

i. Find all critical values of $f(x)$ on $[a, b]$.

ii. Evaluate $f(x)$ at all critical values.

iii. Evaluate $f(x)$ at the endpoints, $f(a)$ and $f(b)$.

iv. The absolute max is the largest function value and the absolute min is the smallest function value.

(l) **L'Hospitals Rule**

i. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \quad , g'(x) \neq 0$$

ii. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

iii. **Indeterminate Form of Type** $0 \cdot \infty$

Given $\lim_{x\to a} f(x)g(x)$, you do the following to put it in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

A. Rewrite $f(x)g(x)$ as $\dfrac{f(x)}{1/g(x)}$

B. Rewrite $f(x)g(x)$ as $\dfrac{g(x)}{1/f(x)}$

iv. **Indeterminate Form of** 0^0**,** ∞^0**, or** 1^∞

Given $\lim_{x\to a} f(x)^{g(x)}$, rewrite the limit as

$$\lim_{x\to a} e^{g(x)\ln(f(x))}$$

then take the limit of the exponent

$$\lim_{x\to a} g(x)\ln(f(x))$$

This should put the limit in the **Indeterminate Form of Type** $0 \cdot \infty$

v. **Indeterminate Form of** $\infty - \infty$

The first thing to try to combine the functions into **One Big Fraction**. Then try L'Hospitals Rule. Another option would be to factor and turn it into a product of two functions.

(m) **Inverse Derivative Theorem**

Let $f(x)$ be a one-to-one function.

$$\left(f^{-1}(b)\right)' = \frac{1}{f'(f^{-1}(b))}$$

4. **Summary of Curve Sketching** – Always start by noting the domain of $f(x)$

   (a) x and y intercepts

      i. x-intercepts occur when $f(x) = 0$

      ii. y-intercept occurs when $x = 0$

   (b) Find any vertical, horizontal asymptotes, or slant asymptotes.

      i. Vertical Asymptote: Find all x-values where $\lim_{x \to a} f(x) = \pm\infty$. Usually when the denominator is 0 and the numerator is not 0. Rational function MUST be reduced.

      ii. Horizontal Aymptotes: Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. There are shortcuts based on the degree of the numerator and denominator.

      iii. Slant Asymptotes: Occurs when the degree of the numerator is one larger than the denominator. You must do long division to determine the asymptotes.

   (c) Find $f'(x)$

      i. Find the critical values, all x-values where $f'(x) = 0$ or when $f'(x)$ does not exist.

      ii. Plot the critical values on a number line.

      iii. Find increasing / decreasing intervals using number line

      iv. Use **The First Derivative Test** to find local maximums / minimums (if any exist). Remember to write them as points.

         A. Local Max at $x = c$ if $f'(x)$ changes from $(+)$ to $(-)$ at $x = c$.

         B. Local Min at $x = c$ if $f'(x)$ changes from $(-)$ to $(+)$ at $x = c$.

         C. Note: If $f'(x)$ does not change signs, it's still an important point. It may be a place where the slope is 0, a corner, an asymptote, a vertical tangent line, etc.

   (d) Find $f''(x)$

      i. Find all x-values where $f''(x) = 0$ or when $f''(x)$ does not exist.

      ii. Plot these x-values on a number line.

      iii. Find intervals of concavity using the number line

      iv. Find points of inflection

         A. Must be a place where concavity changes

         B. The point must exist (i.e, can't be an asymptote, discontinuity)

   (e) Sketch

      i. Draw every asymptote

      ii. Plot all intercepts

      iii. Plot all critical points (even if they are not relative extrema). They were critical points for a reason.

      iv. Plot all inflection points.

      v. Connect points on the graph by using information about increasing/decreasing and its concavity.

5. **Integrals**

   (a) **Definitions**

      i. **Antideriative:** An antiderivative of $f(x)$ is a function $F(x)$, where $F'(x) = f(x)$.

      ii. **General Antiderivative:** The general antiderivative of $f(x)$ is $F(x) + C$, where $F'(x) = f(x)$. Also known as the **Indefinite Integral**

$$\int f(x)\,dx = F(x) + C$$

      iii. **Definite Integral:**

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

   (b) **Approximation Integration Techniques**

   Given the integral $\int_a^b f(x)\,dx$ and n (for Simpson's Rule n must be even), with $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then

   Left-hand $\int_a^b f(x)\,dx = \Delta x\left[f(x_0) + f(x_1) + f(x_2) + f(x_3) + ... + f(x_{n-1})\right]$

   Right-hand $\int_a^b f(x)\,dx = \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + ... + f(x_n)\right]$

   Midpoint $\int_a^b f(x)\,dx = \Delta x\left[f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + ... + f(x_n^*)\right]$, where x_i^* is midpoint in $[x_{i-1}, x_i]$

   Trapezoid $\int_a^b f(x)\,dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$

   Simpson's $\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

   (c) **Common Integrals**

      i. $\int k\,dx = kx + C$

      ii. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

      iii. $\int \frac{1}{x}\,dx = \ln|x| + C$

      iv. $\int \frac{1}{kx+b}\,dx = \frac{1}{k}\ln|kx+b| + C$

      v. $\int e^x\,dx = e^x + C$

      vi. $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$

      vii. $\int e^{kx+b}\,dx = \frac{1}{k}e^{kx+b} + C$

      viii. $\int a^x\,dx = \frac{a^x}{\ln a} + C$

      ix. $\int a^{kx}\,dx = \frac{1}{k\ln a}a^{kx} + C$

      x. $\int a^{kx+b}\,dx = \frac{1}{k\ln a}a^{kx+b} + C$

      xi. $\int \ln x\,dx = x\ln x - x + C$

      xii. $\int \cos x\,dx = \sin x + C$

      xiii. $\int \sin x\,dx = -\cos x + C$

      xiv. $\int \sec^2 x\,dx = \tan x + C$

      xv. $\int \csc^2 x\,dx = -\cot x + C$

      xvi. $\int \sec x\tan x\,dx = \sec x + C$

      xvii. $\int \csc x\cot x\,dx = -\csc x + C$

      xviii. $\int \tan x\,dx = \ln|\sec x| + C$

      xix. $\int \cot x\,dx = \ln|\sin x| + C$

      xx. $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

xxi. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

xxii. $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

xxiii. $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

xxiv. $\int -\frac{1}{a^2 + x^2} \, dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

xxv. $\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + C$

xxvi. $\int -\frac{1}{x\sqrt{x^2 - 1}} \, dx = \csc^{-1}(x) + C$

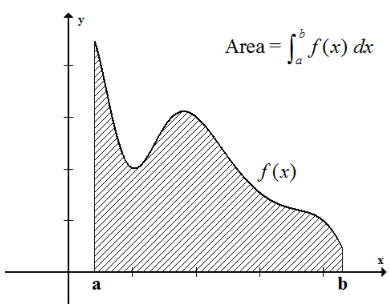
(d) Every integral must be written into the proper form in order to use the formulas. For example,

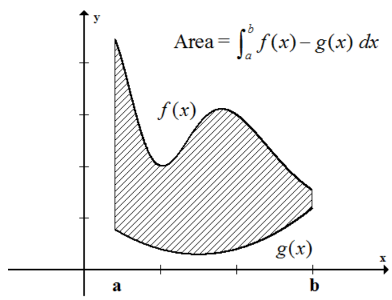
$$\int \sqrt{x} \, dx = \int x^{1/2} \, dx$$

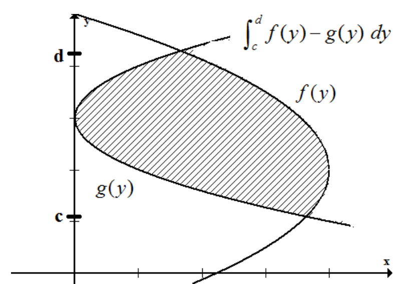
$$\int \frac{3}{5x^4} \, dx = \int \frac{3}{5}x^{-4} \, dx$$

$$\int \frac{8}{\sqrt[5]{x^9}} \, dx = \int 8x^{-9/5} \, dx$$

(e) Finding Area under or between Curves

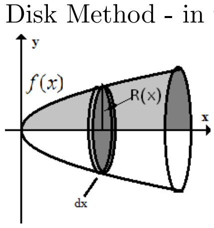






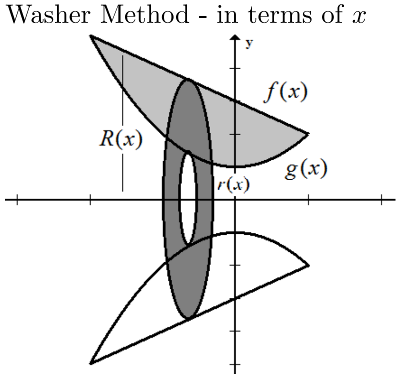
6. **Solids of Revolution**

(a) Disk Method - in terms of x



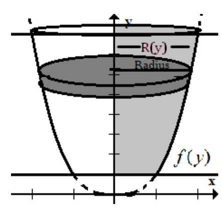
$$V = \int_a^b \pi R(x)^2 \, dx$$

(b) Washer Method - in terms of x



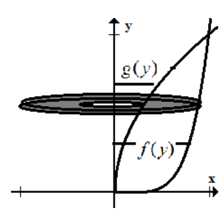
$$V = \int_a^b \pi R(x)^2 - \pi r(x)^2 \, dx$$

(c) Disk Method - in terms of y



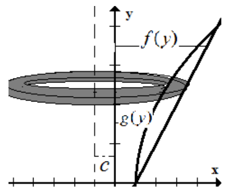
$$V = \int_c^d \pi R(y)^2 \, dy$$

(d) Washer Method - in terms of y



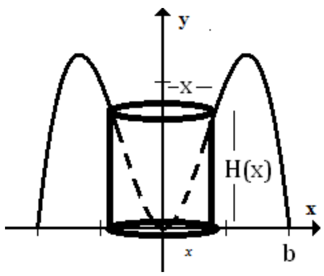
$$V = \int_c^d \pi R(y)^2 - \pi r(y)^2 \, dy$$

You may have to rotate around an axis other than the x or y axis. If you do, you need to adjust the radius. For example,



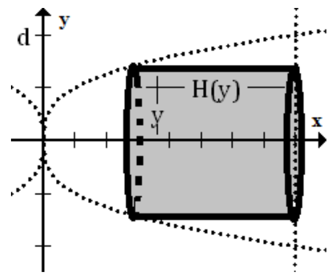
$$V = \int_c^d \pi(1 + R(y))^2 + \pi(1 + r(y))^2 \, dy$$

(a) Shell - in terms of x



$$V = \int_a^b 2\pi x H(x) \, dx$$

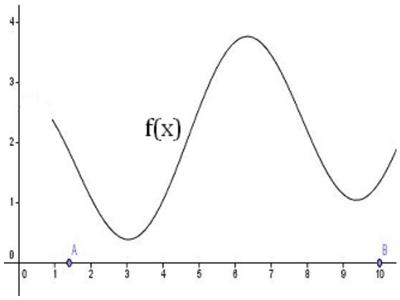
(b) Shell - in terms of y



$$V = \int_c^d 2\pi y H(y) \, dx$$

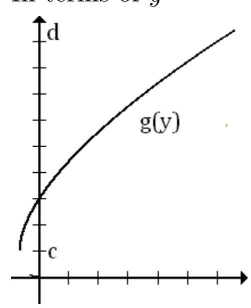
7. Arc Length

(a) In terms of x



$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

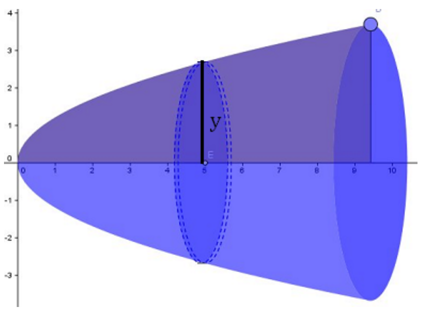
(b) In terms of y



$$L = \int_c^d \sqrt{1 + (g'(x))^2} \, dy$$

8. **Area of a Surface of Revolution**

(a) **Rotated around the x-axis**



$$SA = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad\qquad SA = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The first integral is in terms of x. That means the y in front of the square root should be in terms of x. For example, $y = x^2 + 2$. You use $x^2 + 2$ instead of y.

(b) **Rotated around the y-axis.**

The formula doesn't change much.

$$SA = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad\qquad SA = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

The second integral is in terms of y. That means the x in front of the square root should be in terms of y. For example, if $y = x^2$, solve for x ($x = \sqrt{y}$) and use $\sqrt{y}$ instead of x.

9. **Integration Techniques**

(a) u **Substitution**

Given $\int_a^b f(g(x))g'(x)\, dx$,

i. Let $u = g(x)$

ii. Then $du = g'(x)\, dx$

iii. If there are bounds, you must change them using $u = g(b)$ and $u = g(a)$

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

(b) **Integration By Parts**

$$\int u\, dv = uv - \int v\, du$$

Example: $\int x^2 e^{-x}\, dx$

$$u = x^2 \qquad\qquad dv = e^{-x}\, dx$$
$$du = 2x\, dx \qquad\qquad v = -e^{-x}$$

$$\int x^2 e^{-x}\, dx = -x^2 e^{-x} - \int -2x e^{-x}\, dx$$

You may have to do integration by parts more than once. When trying to figure out what to choose for u, you can follow this guide: LIATE

L Logs

I Inverse Trig Functions

A Algebraic (radicals, rational functions, polynomials)

T Trig Functions ($\sin x$, $\cos x$)

E Exponential Functions

(c) **Products of Trig Functions**

i. $\int \sin^n x \cos^m x \, dx$

A. m is odd (power of $\cos x$ is odd). Factor out one $\cos x$ and place it in front of dx. Rewrite all remaining $\cos x$ as $\sin x$ by using $\cos^2 x = 1 - \sin^2 x$. Then let $u = \sin x$ and $du = \cos x \, dx$

B. n is odd (power of $\sin x$ is odd). Factor out one $\sin x$ and place it in front of dx. Rewrite all remaining $\sin x$ as $\cos x$ by using $\sin^2 x = 1 - \cos^2 x$. Then let $u = \cos x$ and $du = -\sin x \, dx$

C. If n and m are both odd, you can choose either of the previous methods.

D. If n and m are even, use the following trig identities

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$
$$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$$
$$\sin(2x) = 2 \cos x \sin x$$

ii. $\int \tan^n x \sec^m x \, dx$

A. m is even (power of $\sec x$ is even). Factor out one $\sec^2 x$ and place it in front of dx. Rewrite all remaining $\sec x$ as $\tan x$ by using $\sec^2 x = 1 + \tan^2 x$. Then let $u = \tan x$ and $du = \sec^2 x \, dx$

B. n is odd (power of $\tan x$ is odd). Factor out one $\sec x \tan x$ and place it in front of dx. Rewrite all remaining $\tan x$ as $\sec x$ by using $\tan^2 x = \sec^2 x - 1$. Then let $u = \sec x$ and $du = \sec x \tan x \, dx$

C. If n odd and m is even, you can use either of the previous methods.

D. If n is even and m is odd, the previous methods will not work. You can try to simplify or rewrite the integrals. You may try other methods.

(d) **Partial Fractions**

Use this method when you are integrating $\int \frac{p(x)}{q(x)} \, dx$, the degree of $p(x)$ must be smaller than the degree of $q(x)$. Factor the denominator $q(x)$ into a product of linear and quadratic factors. There are four scenarios.

Unique Linear Factors: If your denominator has unique linear factors

$$\frac{x}{(2x-1)(x-3)} = \frac{A}{2x-1} + \frac{B}{x-3}$$

To solve for A and B, multiply through by the common denominator to get

$$x = A(x-3) + B(2x-1)$$

You can find A by plugging in $x = \frac{1}{2}$. Find B by plugging in $x = 3$.

Repeated Linear Factors: Every power of the linear factor gets its own fraction (up to the highest power).

$$\frac{x}{(x-3)(3x+4)^3} = \frac{A}{x-3} + \frac{B}{3x+4} + \frac{C}{(3x+4)^2} + \frac{D}{(3x+4)^3}$$

Unique Quadratic Factor:

$$\frac{3x-1}{(x+4)(x^2+9)} = \frac{A}{x+4} + \frac{Bx+C}{x^2+9}$$

To solve or A, B, and C, multiply through by the common denominator to get

$$3x - 1 = A(x^2+9) + (Bx+C)(x+4)$$

Repeated Quadratic Factor: Every power of the quadratic factor gets its own fraction (up to the highest power).

$$\frac{2x}{(3x+4)(x^2+9)^3} = \frac{A}{3x+4} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2} + \frac{Fx+G}{(x^2+9)^3}$$

(e) **Trig Substitution**

If you see	Substitute	Uses the following Identity
$\sqrt{a^2-x^2}$	$x = a\sin(\theta)$	$1-\sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta)$	$1+\tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta)$	$\sec^2(\theta)-1 = \tan^2(\theta)$

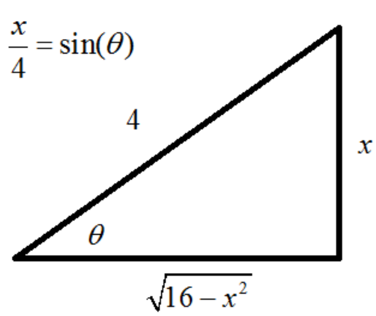
Example: $\int \frac{x^3}{\sqrt{16-x^2}}\,dx$ Let $x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$. So $x^3 = 64\sin^3\theta$

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \int \frac{64\sin^3\theta}{\sqrt{16-16\sin^2\theta}} \cdot 4\cos\theta\,d\theta = \int \frac{64\sin^3\theta \cdot 4\cos\theta d\theta}{4\cos\theta} = \int \sin^3\theta\,d\theta$$

Now you have an integral containing powers of trig functions. You can refer to that method to solve the rest of this integral.

$$\int \sin^3\ d\theta = \frac{\cos^3\theta}{3} - \cos\theta + C$$

To get back to x, we need to use a right triangle with the original substitution $x = 4\sin x$.



If $\cos\theta = \frac{\sqrt{16-x^2}}{4}$, then

$$\int \frac{x^3}{\sqrt{16-x^2}}\,dx = \frac{\left(\frac{\sqrt{16-x^2}}{4}\right)^3}{3} - \frac{\sqrt{16-x^2}}{4} + C$$

10. **Improper Integrals**

(a) Infinity as a bound

i. $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$

ii. $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$

iii. $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

(b) Discontinuity at a bound

i. Discont. at a: $\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$

ii. Discont. at b: $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$

iii. Discont. at c, $a < c < b$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

11. **Sequences and Series**

(a) **Sequences:** A list of numbers in a definite order, $\{a_0, a_1, a_2, a_3, a_4, ...\}$

i. **Sequeeze Theorem** If $a_n < b_n < c_n$, for $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$

ii. **Useful Theorem**

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

iii. **Geometric Sequence**

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \\ \text{Diverge,} & \text{elsewhere} \end{cases}$$

iv. **A typical heirarchy of sequences**

$$C < \ln n < n^p < a^n < n! < n^n$$

(b) **Series**

i. **Definition**

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

ii. **Partial Sums**

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + ... + a_N$$

iii. **Geometric Series:** If $-1 < r < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$\sum_{n=k}^{\infty} ar^{n-k} = \frac{a}{1-r}$$

12. **Summary of Convergence Tests**

Test	Theorem	Comments
Divergence Test	Given the series $\sum a_n$, if $\lim\limits_{n\to\infty} a_n \neq 0$, the series diverges.	Can only be used to test for divergence. It can never be used to test for convergence.
P-Test	The series $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. If $p \leq 1$, then $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ diverges.	You're normally using this series as a comparison for Direct or Limit Comparison Tests
Integral Test	Given the series $\sum a_n$, if you can write $a_n = f(n)$ and $\int_1^{\infty} f(n)\, dn$ converges, then so does $\sum a_n$. If $\int_1^{\infty} f(n)\, dn$ diverges, so does $\sum a_n$	The function f should be easily integrated using one of our integration techniques from previous chapters. This test should not be used when a_n has factorials.
Direct Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series such that $0 \leq a_n \leq b_n$. If $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.	You need to be able to anticipate the convergence. This allows you to choose an appropriate comparison. You should know the convergence of one of the series.
Limit Comparison Test	Let $\sum a_n$ and $\sum b_n$ be series where $a_n \geq 0$ and $b_n \geq 0$. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive finite number, then either both series converge or they both diverge.	Can be easier than Direct Comparison Test since getting the required inequalities to work can be time consuming.
Alternating Series Test	Given the series $\sum (-1)^n b_n$, if $b_n > 0$, decreasing, and $b_n \to 0$, then the series converges.	You have to check for absolute or conditional convergence by testing $\sum \lvert(-1)^n b_n\rvert = \sum b_n$.
Ratio Test	Let $\sum a_n$ be a series. Let $L = \lim\limits_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$. If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges. If $L = 1$, the test is inconclusive.	Use this test when you have exponential functions or factorials.
Root Test	Let $\sum a_n$ be a series. Let $L = \lim\limits_{n\to\infty} \sqrt[n]{\lvert a_n\rvert}$. If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges. If $L = 1$, the test is inconclusive.	Use this test when $a_n = (b_n)^n$.
Absolute Convergence	If the series $\sum a_n$ converges absolutely, then it converges.	If $\sum \lvert a_n\rvert$ converges, when $\sum a_n$ is absolutely convergent.
Conditional Convergence	If the series $\sum a_n$ converges but $\sum \lvert a_n\rvert$ diverges, then the series $\sum a_n$ converges conditionally.	You're usually checking if $\sum \lvert a_n\rvert$ converges because $\sum a_n$ converged by AST

13. **Estimating a Series**

(a) **Remainder Estimate for Integral Test:** If $\sum_{n=1} a_n$ is convergent, $f(n)$ is continuous, positive, and decreasing, then

$$R_N \leq \int_N^\infty f(n)\, dn$$

If you're asked to estimate $\sum_{n=1} a_n$ to 4 decimals (or have an error less than 0.0001), then solve integrate and solve for N.

$$\int_N^\infty f(n)\, dn < 0.0001$$

(b) **Alternating Series Estimation Theorem:** Given a convergent alternating series $\sum_{n=1} a_n = \sum_{n=1} (-1)^n b_n$, then

$$|R_N| \leq b_{N+1}$$

14. **Power Series**

(a) A power series centered at 0

$$\sum_{n=0} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$$

(b) A power series centered at a

$$\sum_{n=0} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ...$$

(c) **Radius and Interval of Convergence:** Perform the Ratio Test or Root Test (usually Ratio Test) For a given a power series $\sum_{n=1}^\infty c_n (x-a)^n$, there are only three possibilities:

i. The series converges only when $x = a$. The interval of convergence is $\{a\}$ and the radius of convergence is $R = 0$. This happens when the Ratio Test gives $L > 1$.

ii. The series converges for all x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$. This happens when the Ratio Test gives $L = 0$.

iii. The series converges on an interval $(a-R, a+R)$. This happens when the Ratio Test gives $K|x-a|$, where you need to solve $K|x-a| < 1$.

15. **Writing Functions as a Power Series**

Given $f(x) = \sum_{n=0}^\infty c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + ...$ with a radius of convergence $R > 0$, then

$$f'(x) = \sum_{n=1}^\infty n c_n (x-a)^{n-1} = 1c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3$$

$$\int f(x)\, dx = \sum_{n=0}^\infty \frac{c_n (x-a)^{n+1}}{n+1} = c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \frac{1}{3}c_2(x-a)^3 + \frac{1}{4}c_3(x-a)^4 + ...$$

You use this technique if you're asked to find a power series of a function like $\frac{1}{(1+x)^2}$, $\ln(1+x)$, $\tan^{-1} x$, etc. Your goal is to differentiate or integrate your function $f(x)$ until it's in the proper form

$$\frac{A}{1-u}$$

A and u can be anything. Once it's in the proper form, you can write it as a power series.

$$\frac{A}{1-u} = \sum_{n=0} A \cdot (u)^n$$

(a) If you had to integrate to get the proper form, then differentiate the power series to get back to the original $f(x)$.

(b) If you differentiated to get the proper form, then integrate the power series to get back to the original $f(x)$.

16. **Taylor Series / Macluarin Series**

(a) Taylor Series is centered at a

$$f(x) = \sum_{n=0} c_n(x-a)^n = c_0+c_1(x-a)+c_2(x-a)^2+...$$

with $c_n = \dfrac{f^n(a)}{n!}$

(b) Maclaurin Series is centered at 0

$$f(x) = \sum_{n=0} c_n(x)^n = c_0 + c_1x + c_2x^2 + ...$$

with $c_n = \dfrac{f^n(a)}{n!}$

It may helfpul to determine the coefficients $c_0, c_1, c_2, ...$ by using the table

n	$f^n(x)$	$f^n(a)$	$c_n = \frac{f^n(a)}{n!}$
0	$f(x)$	$f(a)$	$c_0 = \frac{f(a)}{0!}$
1	$f'(x)$	$f'(a)$	$c_1 = \frac{f'(a)}{1!}$
2	$f''(x)$	$f''(a)$	$c_2 = \frac{f''(a)}{2!}$
3	$f'''(x)$	$f'''(a)$	$c_3 = \frac{f'''(a)}{3!}$

(a) If you need to find the Taylor / Maclaurin Series, find a pattern for c_n. Then you write the following series with the appropriate c_n

$$f(x) = \sum_{n=0} c_n(x-a)^n$$

(b) If you need to find the n-th degree Taylor / Maclaurin Polynomial, then just find the coefficients you need. For example, a 3rd degree Taylor Polynomial requires the coefficients up to c_3.

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3$$