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Math 229
Mock Exam 2

Disclaimer: This mock exam is for practice purposes only. It may not represent your instructor's exam. Doing well on this exam does not guarantee success on your real exam. It also doesn't guarantee failure. Use this exam to find your strengths and weaknesses and to see how long it takes to do certain problems. One of them main obstacles of any calculus exam is time management.

This exam may not be exhaustive. Ask your instructor for the exact sections being covered.

Good luck!

Calculus Tutoring Center
DU 326

Name: SOLUTION KEY

Show all work to receive full credit.

1. Compute the following derivatives.

(a) $y = \dfrac{5}{\sqrt[3]{x}} + 15x - x^{2/3} - \dfrac{1}{8x^8} + 6$

Rewrite y so each term is of the form x^n.

$$y = 5x^{-1/3} + 15x - x^{2/3} - \frac{1}{8}x^{-8} + 6$$

Now use the power rule to differentiate each term.

$$y' = -\frac{5}{3}x^{-4/3} + 15 - \frac{2}{3}x^{-1/3} + x^{-9}$$

(b) $y = \dfrac{8}{(x^3 - x^2 + 1)^{5/4}}$

You can use the quotient rule straightaway or rewrite this so you can use the chain rule. Personally, I'd use the chain rule. So let's rewrite y

$$y = 8(x^3 - x^2 + 1)^{-5/4}$$

Now use the chain rule to differentiate y.

$$y' = 8 \cdot -\frac{5}{4}(x^3 - x^2 + 1)^{-9/4} \cdot (3x^2 - 2x)$$

$$y' = -10(x^3 - x^2 + 1)^{-9/4} \cdot (3x^2 - 2x)$$

$$y' = \frac{-10(3x^2 - 2x)}{(x^3 - x^2 + 1)^{9/4}}$$

(c) $y = \tan^2(\sqrt{x})$

You need to practice rewriting trig functions that have a power. In this problem, you want to rewrite this as

$$y = \left(\tan(\sqrt{x})\right)^2$$

But we also have $\sqrt{x}$, which we should rewrite as $x^{1/2}$. So our function now looks like

$$y = \left(\tan(x^{1/2})\right)^2$$

I'll show all the steps for the chain rule.

$$\begin{aligned}
y' &= \frac{d}{dx}\left[\left(\tan(x^{1/2})\right)^2\right] \\
&= 2\left(\tan(x^{1/2})\right)^1 \cdot \frac{d}{dx}\left[\tan(x^{1/2})\right] \\
&= 2\left(\tan(x^{1/2})\right) \cdot \sec^2(x^{1/2}) \cdot \frac{d}{dx}\left[x^{1/2}\right] \\
&= 2\left(\tan(x^{1/2})\right) \cdot \sec^2(x^{1/2}) \cdot \frac{1}{2}x^{-1/2} \\
&= x^{-1/2} \cdot \tan(x^{1/2}) \cdot \sec^2(x^{1/2})
\end{aligned}$$

(d) $y = \dfrac{2x^3 - \dfrac{5}{x}}{x^{4/3} + x}$

Make sure all the terms are in the right form.

$$y = \frac{2x^3 - 5x^{-1}}{x^{4/3} + x}$$

Now we use the quotient rule.

$$\begin{aligned}
y' &= \frac{(x^{4/3} + x) \cdot \frac{d}{dx}(2x^3 - 5x^{-1}) - (2x^3 - 5x^{-1}) \cdot \frac{d}{dx}(x^{4/3} + x)}{(x^{4/3} + x)^2} \\
&= \frac{(x^{4/3} + x) \cdot (6x^2 + 5x^{-2}) - (2x^3 - 5x^{-1}) \cdot (\frac{4}{3}x^{1/3} + 1)}{(x^{4/3} + x)^2}
\end{aligned}$$

Honestly, there isn't much simplifying here that would make this look that much better. Ask your instructor ahead of time how far he or she wants problems simplified. Since my main objective for this problem is to test your quotient rule skills, I'm going to stop here.

2. Find all x-coordinates where the following function has a horizontal tangent line.

$$y = (3x-1)^4(2x+1)^{-3}$$

(a) Horizontal tangent lines have a slope of 0.

(b) Let's find y' and set it equal to 0.

(c) We differentiate this by using the product rule.

$$\begin{aligned} y' &= (3x-1)^4 \cdot \frac{d}{dx}\left[(2x+1)^{-3}\right] + (2x+1)^{-3} \cdot \frac{d}{dx}\left[(3x-1)^4\right] \\ &= (3x-1)^4 \cdot -3(2x+1)^{-4} \cdot 2 + (2x+1)^{-3} \cdot 4(3x-1)^3 \cdot 3 \\ &= -6(3x-1)^4(2x+1)^{-4} + 12(3x-1)^3(2x+1)^{-3} \end{aligned}$$

At this point, if we want to set $y' = 0$, we need to factor y'. There are two main terms, and they have common factors.

The common factors are 6, $(3x-1)^3$, and $(2x+1)^{-4}$. Note: You pull out (factor) the lowest number they have in common. If the exponents are negative, you still take the smaller number ($-4 < -3$). That's why I pulled out $(2x+1)^{-4}$

$$\begin{aligned} y' &= -6(3x-1)^4(2x+1)^{-4} + 12(3x-1)^3(2x+1)^{-3} \\ &= 6(3x-1)^3 \cdot (2x+1)^{-4} \cdot [-(3x-1) + 2(2x+1)] \\ &= 6(3x-1)^3 \cdot (2x+1)^{-4} \cdot (x+3) \end{aligned}$$

(d) Set $y' = 6(3x-1)^3(2x+1)^{-4}\,(x+3) = 0$, and we get

$$x = -3, \frac{1}{3}$$

.

Note: We don't set $(2x+1)^{-4} = 0$ because it won't make $y' = 0$, it would make it undefined.

3. Find the equation of the tangent line on the curve

$$\sin(x+y) = 2x - 2y$$

at the point (π, π). Leave it in point-slope form.

We use implicit differentiation.

$$\frac{d}{dx}\left[\sin(x+y)\right] = \frac{d}{dx}\left[2x - 2y\right]$$

$$\cos(x+y) \cdot \left(1 + y'\right) = 2 - 2 \cdot y'$$

Solve for y'

$$\cos(x+y) + \cos(x+y) \cdot y' = 2 - 2 \cdot y'$$

$$\cos(x+y) \cdot y' + 2 \cdot y' = 2 - \cos(x+y)$$

$$y'\left(\cos(x+y) + 2\right) = 2 - \cos(x+y)$$

$$y' = \frac{2 - \cos(x+y)}{\cos(x+y) + 2}$$

To find the slope of the tangent line at (π, π), plug in $x = \pi$ and $y = \pi$ into y'

$$y'(\pi, \pi) = \frac{2 - \cos(\pi + \pi)}{\cos(\pi + \pi) + 2}$$

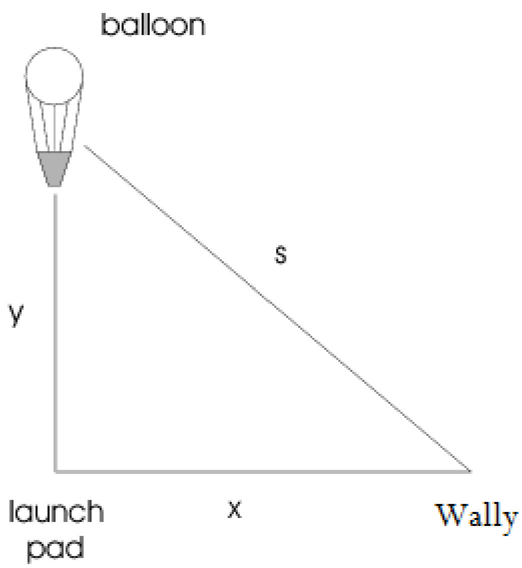
$$y' = \frac{1}{3} \text{ at } (\pi, \pi)$$

The equation of the tangent line is

$$y - \pi = \frac{1}{3}\left(x - \pi\right)$$

4. Carrie and David elope in a hot air balloon, which rises at a constant rate of 5 meters per second. Just as the hot air balloon begins to rise, Carrie's other potential suitor Wally arrives to stop them. He parks 64 meters from the launch pad, and runs towards the pad at 4 meters per second. At what rate is the distance between Wally and the balloon changing when the balloon is 30 meters above the ground.

Here's a picture to get you started.



(a) The first step in most related rates problems is to draw a picture. I've done this for you already. It is extremely unlikely this would be given to you on the test.

(b) What do we know?

- Wally's distance from the launchpad is x. Since x is decreasing as Wally runs to it, his rate is $\frac{dx}{dt} = -4$
- $\frac{dy}{dt} = 5$
- s is the distance from Wally to the balloon

(c) What we want?

- When $y = 30$, what is $\frac{ds}{dt}$?

(d) We need a formula that relates all these variables together. So what formula relates x, y, and s, (three sides of a right triangle) together?

$$x^2 + y^2 = s^2$$

(e) Differentiate this

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2s \cdot \frac{ds}{dt}$$

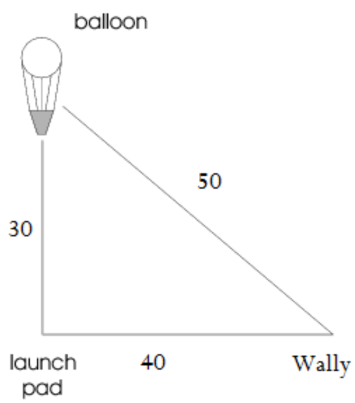
Cancel the 2s

$$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = s \cdot \frac{ds}{dt}$$

(f) Now we can plug in all our specific information, such as $y = 30$. But if $y = 30$, what is x and s?

Since the balloon is 30 meters high, 6 seconds has passed. In 6 seconds, Wally ran $6 \cdot 4 = 24$ meters. Note that $x \neq 24$. Remember that x represents the distance from the launchpad. He starts 64 meters away and runs 24 meters. That means Wally's distance from the launchpad is $x = 40$ meters. Now let's find s.

We use the pythagorean theorem to find s.



$$30^2 + 40^2 = s^2$$

$$s = 50 \text{ meters}$$

(g) Now plug everything in to find $\frac{ds}{dt}$

$$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = s \cdot \frac{ds}{dt}$$

$$40 \cdot -4 + 30 \cdot 5 = 50 \cdot \frac{ds}{dt}$$

$$-10 = 50 \cdot \frac{ds}{dt}$$

$$\frac{ds}{dt} = -\frac{1}{5} \text{ m/s}$$

5. At this point, everyone should know what $\tan(\pi/4)$ equals. Well, I want you to estimate $\tan(3\pi/11)$ by following these two easy steps.

(a) Find the formula for $L(x)$, the linearization of $f(x) = \tan(x)$ near $a = \pi/4$.

The formula for $L(x)$ is $L(x) = f(a) + f'(a)(x - a)$. So we need

- a

$$a = \pi/4$$

- $f(a)$

$$\tan(\pi/4) = 1$$

- $f'(a)$

$$f'(x) = \sec^2(x)$$

So

$$f'(\pi/4) = \sec^2(\pi/4) = 2$$

Therefore, $L(x) = 1 + 2(x - \pi/4)$

(b) Use $L(x)$ to approximate $\tan(3\pi/11)$. Do not round your answer.

To estimate $\tan(3\pi/11)$, we need to know what to plug in to $f(x)$ to get $\tan(3\pi/11)$. Most of the time it's easy. We have to plug in $x = 3\pi/11$. Note: sometimes, it won't be that easy. Anyhow, let's use the linearization to estimate $\tan(3\pi/11)$

$$\tan(3\pi/11) \approx L(3\pi/11) = 1 + 2(3\pi/11 - \pi/4)$$

$$= 1 + 2(\pi/44)$$

$$= 1 + \pi/22$$

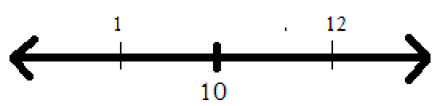
6. Brian manufactures and sells super crazy straws. The price p of selling these crazy straws determines the revenue. The revenue function $R(p)$ has been determined to be $R(p) = -5p^2 + 100p$.

(a) Determine the price at which revenue is at its max?

Recall that a if a function is going to have a local max or min, it must occur when $R'(p) = 0$ or when $R'(p)$ does not exist.

$$R'(p) = -10p + 100$$

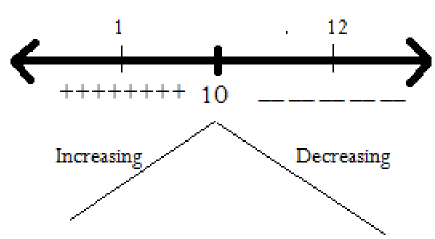
Setting $R'(p) = 0$, we get $p = 10$. This does not guarantee a max revenue, only that it's possible. We use the number line to verify if it's a max.



- Choose a number from each region. I choose $p = 1$ and $p = 12$.
- Plug them in to the derivative function $R'(p)$. This will tell us if the revenue is increasing or decreasing.

$$p = 1 \rightarrow R'(1) = 90$$
$$p = 12 \rightarrow R'(12) = -20$$

- This means the revenue function is increasing on the interval (0,10). When the price is greater than \$10, the revenue function is decreasing. This verifies that the price at which revenue is a max is when $p = \$10$.



(b) What is the max revenue?

To find the max revenue, we just plug $p = 10$ into the revenue function. We just showed in part (a), that the max revenue must occur when $p = 10$.

$$R(10) = -5(10)^2 + 100(10) = 500$$

The max revenue is \$500.