MATH 229, Fall 2012, Holland, Exam II NAME Solution

1. (16 points) Find $\frac{dy}{dx}$. You do NOT need to simplify on this page.

(a) $y = \sec(\cos(\pi x))$ Chain Rule

$$y' = \sec(\cos(\pi x)) \cdot \tan(\cos(\pi x)) \cdot -\sin(\pi x) \cdot \pi$$

(b) $y = \dfrac{\sin(5x)}{(4x+5)^3}$ Quotient

$$y' = \frac{(4x+5)^3 \cdot \cos(5x) \cdot 5 - \sin(5x) \cdot 3(4x+5)^2 \cdot 4}{(4x+5)^6}$$

(c) $y = \tan(x^3)\tan(x^5)\tan(x^7)$ Use $\frac{d}{dx}[fgh] = f'gh + fg'h + fgh'$ if you remember it

$$y' = \underbrace{(\sec^2(x^3) \cdot 3x^2)}_{f'}\underbrace{\tan(x^5)}_{g}\underbrace{\tan(x^7)}_{h} + \underbrace{\tan(x^3)}_{f}\underbrace{(\sec^2(x^5) \cdot 5x^4)}_{g'}\underbrace{\tan(x^7)}_{h}$$
$$+ \underbrace{\tan(x^3)}_{f}\underbrace{\tan(x^5)}_{g}\underbrace{(\sec^2(x^7) \cdot 7x^6)}_{h'}$$

(d) $y = \sin^4(\sqrt{x}\tan(x))$

$$y = (\sin(x^{1/2}\tan x))^4$$

$$y' = 4(\sin(x^{1/2}\tan x))^3 \cdot \cos(x^{1/2}\tan x) \cdot (x^{1/2}\sec^2 x + \tfrac{1}{2}x^{-1/2} \cdot \tan x)$$

2. (16 points) In each case, differentiate, then find all x such that $f'(x) = 0$.

(a) $f(x) = (3x-4)^2\sqrt{5x-1} = (3x-4)^2(5x-1)^{1/2}$

$$f'(x) = (3x-4)^2 \cdot \tfrac{1}{2}(5x-1)^{-1/2} \cdot 5 + 2(3x-4)\cdot 3 \cdot (5x-1)^{1/2}$$
$$= \tfrac{5}{2}(3x-4)^2(5x-1)^{-1/2} + 6(3x-4)(5x-1)^{1/2}$$
$$= (3x-4)(5x-1)^{-1/2}\left(\tfrac{5}{2}(3x-4) + 6(5x-1)\right)$$
$$= (3x-4)(5x-1)^{-1/2}\left(\tfrac{15}{2}x - 10 + 30x - 6\right)$$
$$= (3x-4)(5x-1)^{-1/2}\left(\tfrac{75}{2}x - 16\right)$$

$f'(x) = 0$ when $(3x-4) = 0$, $\left(\tfrac{75}{2}x - 16\right) = 0$

$x = 4/3$ $\qquad$ $x = \dfrac{32}{75}$

Note: $(5x-1)^{-1/2} \neq 0$

(b) $f(x) = \dfrac{3}{(2x-7)^4} - \dfrac{1}{(2x-7)^5} + 9$

Rewrite as $f(x) = 3(2x-7)^{-4} - (2x-7)^{-5} + 9$

$$f'(x) = -12(2x-7)^{-5}\cdot 2 + 5(2x-7)^{-6}\cdot 2$$
$$= -24(2x-7)^{-5} + 10(2x-7)^{-6}$$
$$= -2(2x-7)^{-6}\left(12(2x-7) - 5\right)$$
$$= -2(2x-7)^{-6}\left(24x - 84 - 5\right)$$
$$= -2(2x-7)^{-6}\left(24x - 89\right)$$

$f'(x) = 0$ when $(24x - 89) = 0 \longrightarrow x = \dfrac{89}{24}$

Note: $(2x-7)^{-6} \neq 0$

3. (24 points) Give full answers to the following limit questions, splitting into sides, if necessary.

(a) $\lim_{x\to\pi/4} \frac{3+\cos x}{1-\tan x}$ $\rightarrow \frac{3+\cos(\pi/4)}{1-\tan(\pi/4)} = \frac{3+\sqrt{2}/2}{0}$ ← This means there is a V.A. at $x=\pi/4$.

Right Hand: $\lim_{x\to\pi/4^+} \frac{3+\cos(x)}{1-\tan(x)} \approx \frac{+}{-} = -\infty$

Left Hand: $\lim_{x\to\pi/4^-} \frac{3+\cos(x)}{1-\tan(x)} = \frac{+}{+} = +\infty$

$\lim_{x\to\pi/4} \frac{3+\cos(x)}{1-\tan x}$ Does not exist

(b) $\lim_{x\to 0} \frac{3x^2\sin x + 4\sin x}{x}$ $= \lim_{x\to 0} \frac{3x^2\sin x}{x} + \frac{4\sin x}{x}$

$= \lim_{x\to 0} 3x\sin(x) + \frac{4\sin(x)}{x}$

$= 3(0)\sin(0) + 4\cdot \underline{1}$ ← Recall $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$= 4$

(c) $\lim_{x\to 3}\left(\frac{2}{(x-3)^2} - \frac{3}{x-3} + 4\right)$ we want to focus on $\lim_{x\to 3}\left(\frac{2}{(x-3)^2} - \frac{3}{x-3}\right)$

(1) Combine into one fraction: $\frac{2}{(x-3)^2} - \frac{3}{x-3}\cdot\frac{(x-3)}{(x-3)} = \frac{2}{(x-3)^2} - \frac{3x-9}{(x-3)^2}$

$= \frac{-3x+11}{(x-3)^2}$. Plug in $x=3$, we get $\frac{-9+11}{0} = \frac{2}{0}$

It means there is a V.A. at $x=3$

$\lim_{x\to 3^+} \frac{-3x+11}{(x-3)^2} \rightarrow \frac{+}{+} = +\infty$

$\lim_{x\to 3^-} \frac{-3x+11}{(x-3)^2} \Rightarrow \frac{+}{+} = +\infty$

$\lim_{x\to 3}\left(\frac{2}{(x-3)^2} - \frac{3}{x-3} + 4\right) = \infty + 4$
$= \infty$

4. (10 points) The position of a certain object traveling in a straight line is given by $s(t) = \sin^2(t)$.

(a) Find $a(t)$, the acceleration of the object.

$$v(t) = \frac{d}{dt}\left[\sin^2 t\right] = 2\sin(t)\cos(t)$$

$$a(t) = \frac{d}{dt}\left[2\sin(t)\cos(t)\right] = 2\sin(t)\cdot -\sin(t) + 2\cos(t)\cdot\cos(t)$$

$$= -2\sin^2(t) + 2\cos^2(t)$$

(b) For what value(s) of t in the interval $[0, \pi]$ is the acceleration of the object equal to 0?

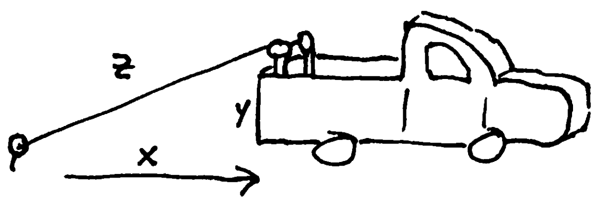
Solve $\quad a(t) = -2\sin^2(t) + 2\cos^2(t) = 0$

$$-2\sin^2(t) = -2\cos^2(t)$$

$$\frac{\sin^2(t)}{\cos^2(t)} = 1 \rightarrow \left(\frac{\sin(t)}{\cos(t)}\right)^2 = 1 \rightarrow \tan^2(t) = 1$$

$$t = \pi/4$$

5. (10 points) A spool of cable is mounted on a truck bed three feet off the ground. The end of the cable is fixed to a hook in the ground, and the truck drives (straight) away from the hook at 3 ft/sec, keeping the cable taut. At what rate is the cable reeling off the spool when the truck is 10 feet from the hook? (Exact answer, may contain a radical.)



Know: $\frac{dx}{dt} = 3$ ft/s

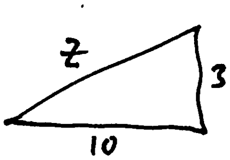
$y = 3$

Find $\frac{dz}{dt}$ when $x = 10$

(1) Formula relating $x, y, \& z$: $\quad x^2 + y^2 = z^2 \longrightarrow x^2 + 3^2 = z^2$

(2) Differentiate: $\quad 2x\cdot\frac{dx}{dt} + 0 = 2z\cdot\frac{dz}{dt} \longrightarrow x\cdot\frac{dx}{dt} = z\cdot\frac{dz}{dt}$

(3) we know $x = 10$, $\frac{dx}{dt} = 3$, but we need to find z. Use Pythagorean's Thm



$$10^2 + 3^2 = z^2$$

$$109 = z^2$$

$$z = \sqrt{109}$$

So $\quad x\cdot\frac{dx}{dt} = z\cdot\frac{dz}{dt}$

$$\Rightarrow 10\cdot 3 = \sqrt{109}\cdot\frac{dz}{dt}$$

$$\Rightarrow \frac{dz}{dt} = \frac{30}{\sqrt{109}} \text{ ft/s}$$

$$\approx 2.87 \text{ ft/s}$$

6. (14 points) Find the **equation** of the line tangent to the curve

$$x^2 + xy - y^2 = 1 \text{ at the point}(2, 3).$$

Use Implicit Differentiation: $2x + x \cdot \frac{dy}{dx} + 1 \cdot y - 2y \cdot \frac{dy}{dx} = 0$

(1) Solve for $\frac{dy}{dx}$: $x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = -2x - y$

$$\frac{dy}{dx}(x - 2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x-2y}$$

(2) Use $y - y_1 = m(x - x_1)$

(3) $m = \frac{dy}{dx}$ at $x = 2, y = 3$

$$m = \frac{-2(2) - 3}{2 - 2(3)} = \frac{-7}{-4} = \frac{7}{4}$$

(4) $y - 3 = \frac{7}{4}(x - 2)$

$y - 3 = \frac{7}{4}x - \frac{7}{2}$

$y = \frac{7}{4}x - \frac{1}{2}$

7. (10 points) Let $f(x) = \frac{1}{x}$.

(a) Use the linearization $L(x)$ of g at $a = 5$ to give a decimal approximation for $\frac{1}{5.02}$.

Use $L(x) = f(a) + f'(a)(x - a)$

(1) $a = 5$

(2) $f(a) = f(5) = \frac{1}{5}$

(3) $f'(x) = \frac{-1}{x^2}$

(4) $f'(5) = \frac{-1}{5^2} = \frac{-1}{25}$

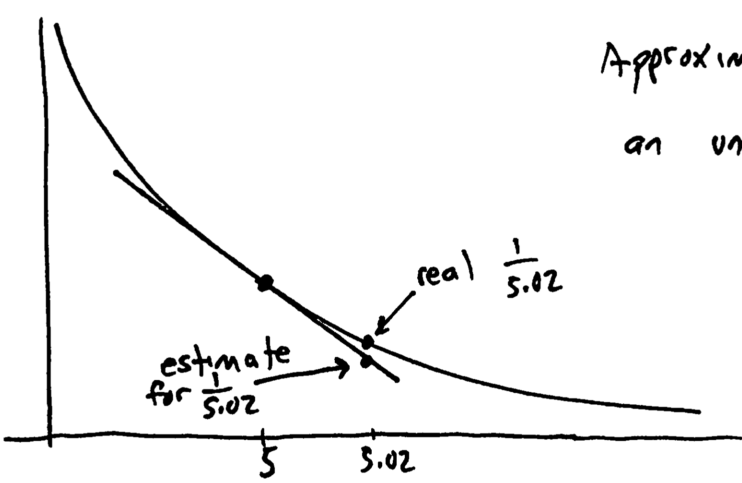
(5) $L(x) = \frac{1}{5} + \frac{-1}{25}(x - 5)$

(6) $\frac{1}{5.02} = f(5.02) \approx L(5.02)$

$L(5.02) = \frac{1}{5} - \frac{1}{25}(5.02 - 5)$

$= 0.1992$

(b) Should your answer be an overestimate of the actual value, or an underestimate? Explain. You may use an appropriate graph to help illustrate your explanation.



Approximation for $\frac{1}{5.02}$ is an underestimate.