1. Carefully state the following theorems, making sure you have the hypotheses correct.

   - **The Extreme Value Theorem**: If f is continuous on a closed interval $[a, b]$, then f has an absolute maximum and absolute minimum somewhere on $[a, b]$.

   - **The Mean Value Theorem**: Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

   - **Fermat's Theorem**: If f has a local minimum or maximum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.

2. From which of the three theorems above can it be argued that if a drain pipe

   lls a 10 gallon bucket in 3 minutes, then at some point during that period the drain was owing at rate of over 150 gallons per hour. Explain

   First, let's convert 150 gallons per hour to gallons per minute.

$$\frac{150 \text{ gallons}}{1 \text{ hour}} \cdot \frac{1 \text{ hour}}{60 \text{ minutes}}$$
$$= \frac{2.5 \text{ gallons}}{1 \text{ minute}}$$

   So when the bucket starts to fill, it has 0 gallons at time 0. Written as a point, that's (0,0). We know after 3 minutes the bucket has 10 gallons. Written as a point, it's (3,10). Finding the slope between these two points, we get

$$\frac{f(3) - f(0)}{3 - 0} = \frac{10}{3}$$

   So the average slope (slope of the secant line) is 3.333 gallons per minute. This means at some point the bucket must be filling up at a rate of 3.333 gallons per minute. This uses the Mean Value Theorem.

   Think of this as your speed driving on a highway. In order to get to 3.333 gallons per minute, you have to (at some point) be going at 2.5 gallons per minute.

3. If Newton's Method, with an initial value of $x_1 = 4$, is used to approximate $\sqrt{8}$, the postive zero of $f(x) = x^2 - 8$, what are x_2 and x_3.

(a) First, we need Newton's Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(b) If $f(x) = x^2 - 8$, then $f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - 8}{2x_n}$$

(c) Use Newton's Method

$$x_2 = x_1 - \frac{x_1^2 - 8}{2x_1^2}$$

$$x_2 = 4 - \frac{4^2 - 8}{2(4)^2}$$

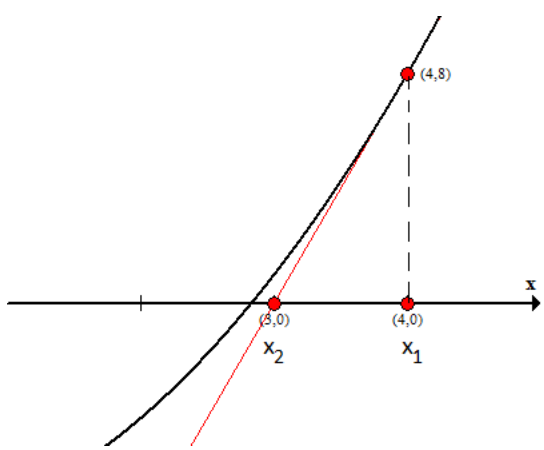
$$x_2 = 3$$

$$x_3 = x_2 - \frac{x_2^2 - 8}{2x_2^2}$$

$$x_3 = 3 - \frac{3^2 - 8}{2(3)^2}$$

$$x_3 = 2.83333$$

(d) Sketch the graph of $f(x)$. Show your graph how Newton's Method constructs x_2 from x_1.



4. Show that the equation $3 - 5x^3 - 6x^5 = 0$ has exactly one solution. State any theorems you use.

(a) To show you have at least one solution, we use the intermediate value theorem.

$$x = 0, \ f(0) = 3$$

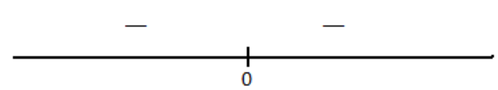
$$x = 1, \ f(1) = -8$$

Since our equation is continuous, the intermediate value theorem states we must have at least one root in the interval (0,1).

(b) Next, we use Rolle's Theorem to determine if we can have two roots. In order to have two roots with a function that is continuous and differentiable, our function needs to turn around. As a function crosses the x-axis (one root), it must turn around to come back to the x-axis to get the 2nd root. This means our function must have a slope of 0 if it has any chance of having a 2nd root.

$$f'(x) = -15x^2 - 30x^4 = -15x^2(1 + 2x^2)$$

It appears the only place we have a slope of 0 is at $x = 0$. Let's use the number line to see if it turns around at $x = 0$.



The function $f(x) = 3-5x^3-6x^5$ does not turn around at $x = 0$. This means our function never turns around. If it never turns around, it can't have a second root. Therefore, our equation has exactly one root.

5. Find the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^{3/2} - 2x^2 + 1}{3x^2 - 5x^3}$.

Using the shortcut I showed you in class, we can do the following

$$\lim_{x\to\infty} \frac{x^{3/2} - 2x^2 + 1}{3x^2 - 5x^3} = \lim_{x\to\infty} \frac{-2x^2}{-5x^3} = \lim_{x\to\infty} \frac{2}{5x} = 0$$

(b) $\displaystyle\lim_{x\to\infty} (3x + 1) \sin\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} (3x + 1) \sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} 3x \sin\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{3\sin(1/x)}{1/x} + \sin(1/x)$$

$$= 3 \cdot 1 + \sin(0)$$

$$= 3$$

If you're confused about what happened to $\frac{\sin(1/x)}{1/x}$, remember we went over limits that had the form $\frac{\sin(0)}{0}$.

(c) $\lim_{x\to 0} \left(x - \sqrt{x^2+3x+4}\right)$

$$\begin{aligned}
\lim_{x\to 0} \left(x - \sqrt{x^2+3x+4}\right) &= \lim_{x\to 0} \left(x - \sqrt{x^2+3x+4}\right) \cdot \frac{x+\sqrt{x^2+3x+4}}{x+\sqrt{x^2+3x+4}} \\
&= \lim_{x\to\infty} \frac{x^2-(x^2+3x+4)}{x+\sqrt{x^2+3x+4}} \\
&= \lim_{x\to\infty} \frac{-3x-4}{x+\sqrt{x^2+3x+4}}
\end{aligned}$$

Note that $x + \sqrt{x^2+3x+4}$ acts like $x + \sqrt{x^2} = x + x = 2x$

$$\lim_{x\to\infty} \frac{-3x-4}{x+\sqrt{x^2+3x+4}} = \lim_{x\to\infty} \frac{-3x}{2x} = \frac{-3}{2}$$

6. Let $f(x) = 11x + \frac{22}{x} - 10$.

(a) Name the theorem that guarantees that f has both an absolute maximum and an absolute minimum over the interval [-4,-1].

The Extreme Value Theorem. The only requirements for this theorem is that the function is continuous on a closed interval. $f(x)$ is continuous on the closed interval [-4,-1].

Some of you might see the $\frac{22}{x}$ and say 'it's undefined at $x = 0$.' This is true but $x = 0$ is not in the interval, so we're safe.

(b) Find the absolute maximum and minimum.

i. We need to find $f'(x)$

$$f'(x) = 11 - \frac{22}{x^2}$$

ii. Next, we set $f'(x) = 0$ and solve.

$$11 - \frac{22}{x^2} = 0$$
$$11x^2 - 22 = 0$$
$$11(x^2 - 2) = 0$$
$$x^2 - 2 = 0$$

The solutions are $x = \pm\sqrt{2}$. The only one in our interval is $-\sqrt{2}$. This is our critical value.

iii. Evaluate $f(x)$ at the endpoints and the critical values in the interva.

$$x = -4, \ f(-4) = -59.5$$
$$x = -1, \ f(-1) = -43$$
$$x = -\sqrt{2} \ f(-\sqrt{2}) = -41.1127$$

The absolute max is $(-\sqrt{2}, -41.1127)$ and the absolute min is $(-4, -59.5)$.

7. Use calculus to find the point on the curve $y = \sqrt{x}$ that is closest to the point $(0, 108)$.

Any point on the curve $y = \sqrt{x}$ has to have the form $(x, \sqrt{x})$. To find which point is closest to (0,108), we want to minimum the distance between (0,108) and $(x, \sqrt{x})$.

$$\text{Distance Formula: } D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$= \sqrt{(0 - x)^2 + (108 - \sqrt{x})^2}$$
$$= \sqrt{x^2 + (108 - \sqrt{x})^2}$$

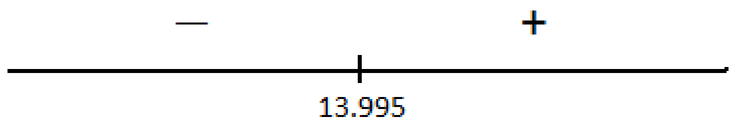
To find the minimum, we find the derivative of the distance function. Now a shortcut is to do the following, **you only have to differentiate the inside**.

$$2x + 2(108 - \sqrt{x}) \cdot -\frac{1}{2\sqrt{x}}$$
$$= 2x - \frac{1}{\sqrt{x}} \cdot (108 - \sqrt{x})$$
$$2x - \frac{108}{\sqrt{x}} + 1$$

Now we set the derivative equal to 0 and solve. I'll leave it to you to check it. The solution to

$$2x - \frac{108}{\sqrt{x}} + 1 = 0$$

is $x = 13.955$. To check if it's a minimum, let's use the numberline.



Plug $x = 13.995$ into $y = \sqrt{x}$ to get the y value for the point. Therefore, the point on $y = \sqrt{x}$ that is closest to $(0, 108)$ is $(13.995, 3.741)$.

8. Let $f(x) = \dfrac{x}{x^3 - 1}$. To do this problem quickly, I'll just give the derivatives without showing work.

   (a) Find any vertical and horizontal asymptotes.

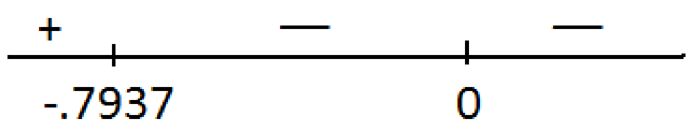
   $$\text{Horizontal: } y = 0$$

   $$\text{Vertical: } x = 1$$

   (b) Find the intevals of increasing and decreasing and find all local extrema.

   $$f'(x) = \frac{-2x^3 - 1}{(x^3 - 1)^2}$$

   The critical values are $x = -.7937$ and $x = 1$. Use the numberline.



   $$\text{Increasing: } (-\infty, -.7937)$$

   $$\text{Decreasing: } (-.7937, 0) \cup (0, \infty)$$

   Local Maximum: (-.7937, .53).

   (c) Find intervals of concavity and any inflection points.

   $$f''(x) = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3}$$

   The 'critical values' to $f''(x)$ are $x = 0$, $x = 1$ and $x = -1.26$. You can check the numberline.

$$\text{Concave Up: } (-\infty, -1.26) \cup (1, \infty)$$

$$\text{Concave Down: } (-1.26, -1)$$

Inflection Points: $(-1.26, .42)$.

(d) Sketch