3.5 Summary of Curve Sketching

Follow these steps to sketch the curve.

1. Domain of $f(x)$
2. x and y intercepts
   (a) x-intercepts occur when $f(x) = 0$
   (b) y-intercept occurs when $x = 0$
3. Symmetry: Is it even or odd or neither. This usually isn't of help.

   If $f(-x) = -f(x)$, then $f(x)$ is symmetric about the origin.

   If $f(-x) = f(x)$, then $f(x)$ is symmetric about the y-axis.
4. Find any vertical or horizontal asymptotes.
   (a) Vertical Asymptote: Find all x-values where $\lim_{x\to a} f(x) = \pm\infty$. Usually when the denominator is 0 and the numerator is not 0
   (b) Horizontal Aymptotes: Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
5. Find $f'(x)$
   (a) Find the critical values, all x-values where $f'(x) = 0$ or when $f'(x)$ does not exist.
   (b) Find increasing / decreasing intervals using numberline
   (c) Find local maximums / minimums (if any exist). Remember to write them as points.
      i. Local Max at $x = c$: $f'(x)$ changes from (+) to (-) at $x = c$.
      ii. Local Min at $x = c$: $f'(x)$ changes from (-) to (+) at $x = c$.
   (d) Plot them
6. Find $f''(x)$
   (a) Find all x-values where $f''(x) = 0$ or when $f''(x)$ does not exist.

(b) Find intervals of concavity using the number line

(c) Find points of inflection

i. Must be a place where concavity changes

ii. The point must exist (i.e, can't be an asymptote, discontinuity)

(d) Plot them

7. Sketch

Example 3.17. Sketch $y = \dfrac{x}{\sqrt{x^2+1}}$

It's probably best to rewrite $f(x)$ as $f(x) = \dfrac{x}{(x^2+1)^{1/2}}$

1. Domain: There are no domain issues.

2. Intercepts:

$$y - \text{intercept: } (0,0)$$

$$x - \text{intercept: } (0,0)$$

3. Symmetry:

$$f(-x) = \frac{(-x)}{\sqrt{(-x)^2+1}} = \frac{-x}{\sqrt{x^2+1}} = -f(x)$$

   This is an odd function. This means once we finish sketching, if the graph is not symmetric about the origin we did something wrong.

4. Asymptotes:

   (a) Horizontal Asymptote:

$$\begin{aligned} \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}} \\ &= \lim_{x\to\infty} \frac{x}{x} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}} \\ &= \lim_{x\to\infty} \frac{x}{-x} \\ &= -1 \end{aligned}$$

   Recall: $\sqrt{x^2} = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

(b) Vertical Asymptote:

There are none. Nothing makes $\sqrt{x^2+1} = 0$.

5. Find y'. Let's use the Quotient Rule.

$$\begin{aligned} y' &= \frac{(x^2+1)^{1/2} \cdot (1) - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{[(x^2+1)^{1/2}]^2} \\ &= \frac{(x^2+1)^{1/2} - x^2(x^2+1)^{-1/2}}{x^2+1} \\ &= \frac{(x^2+1)^{-1/2}[(x^2+1) - x^2]}{x^2+1} \\ &= \frac{(x^2+1)^{-1/2}}{x^2+1} \\ &= \frac{1}{(x^2+1)^{3/2}} \end{aligned}$$

Critical Values: There are no critical values.

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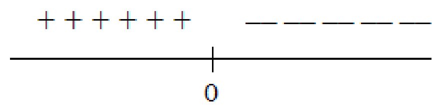
Since there are no critical values. Check any point in the domain, the graph is increasing the entire time.

Increasing: $(-\infty, \infty)$

6. Find y'': Rewrite $y' = (x^2+1)^{-3/2}$

$$\begin{aligned} y'' &= -\frac{3}{2}(x^2+1)^{-5/2} \cdot (2x) \\ &= \frac{-3x}{(x^2+1)^{5/2}} \end{aligned}$$

Critical Values: $x = 0$

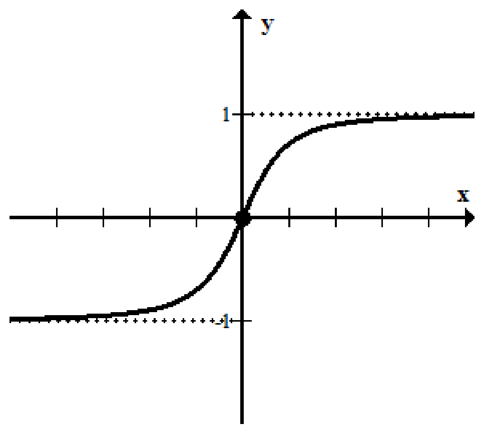


Concave Up: $(-\infty, 0)$

Concave Down: $(0, \infty)$

Point of Inflection: $(0, 0)$

7. Sketch the graph



Example 3.18. Sketch $y = 2\sqrt{x} - x$

1. Domain: $x \geq 0$

2. Intercepts:

   (a) y-intercept: $(0, 0)$

   (b) x-intercepts: Set $2\sqrt{x} - x = 0$

$$\begin{aligned} 2\sqrt{x} - x &= 0 \\ 2\sqrt{x} &= x \\ (2\sqrt{x})^2 &= (x)^2 \\ 4x &= x^2 \\ 0 &= x^2 - 4x \\ 0 &= x(x-4) \end{aligned}$$

which gives us x-intercepts $(0, 0)$ and $(4, 0)$.

3. There is no symmetry.

4. Asymptotes:

   (a) There are no vertical asymptotes.

   (b) There are no horizontal asymptotes.

$$\lim_{x\to\infty} 2\sqrt{x} - x = -\infty$$

We already did an example of this type of limit. Use that technique to show it's $-\infty$.

5. Find y'

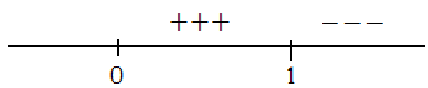
$$\begin{aligned} y' &= 2 \cdot \frac{1}{2} x^{-1/2} - 1 \\ &= x^{-1/2} - 1 \\ &= \frac{1}{\sqrt{x}} - 1 \end{aligned}$$

We have a critical value at $x = 0$ because y' does not exist when $x = 0$. Now we need to solve $y' = 0$

$$\begin{aligned} \frac{1}{\sqrt{x}} - 1 &= 0 \\ \frac{1}{\sqrt{x}} &= 1 \\ 1 &= \sqrt{x} \\ 1 &= x \end{aligned}$$

The other critical value is at $x = 1$.

6. Use the number line to determine where y is increasing or decreasing.



Note, we did not have to pick a number in the region less than 0 since that region is not in the domain.

Increasing: $(0, 1)$

Decreasing: $(1, \infty)$

Local Maximum: $(1, 1)$

Local Minimum: None

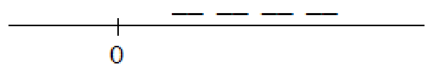
7. Find y''. First, rewrite y' as $y' = x^{-1/2} - 1$.

$$y'' = -\frac{1}{2}x^{-3/2}$$

$$y'' = -\frac{1}{2x^{3/2}}$$

There is a critical value at $x = 0$.

8. Use the number line to determine concavity.

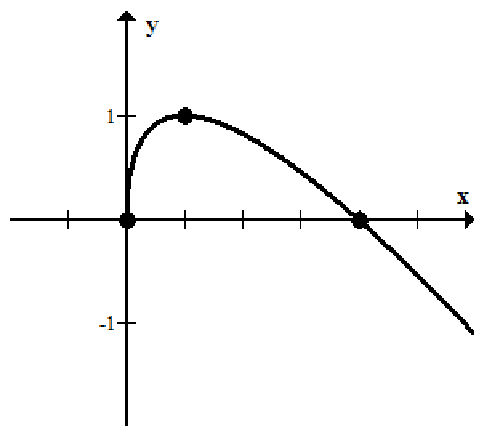


Concave Down: $(0, \infty)$

Concave Up: Never

No points of inflection

9. Sketch the graph



Example 3.19. Sketch $y = 1 + \frac{1}{x} + \frac{1}{x^2}$

1. Domain: $x \neq 0$

2. Intercepts:

   (a) y-intercept: None, because $x \neq 0$

   (b) x-intercept(s): Let's set $y = 0$

$$1 + \frac{1}{x} + \frac{1}{x^2} = 0$$

Multiply through by x^2

$$x^2 + x + 1 = 0$$

There are no solutions to this equation. This means y does not have any x-intercepts.

3. There is no symmetry.

4. Asymptotes:

   (a) Vertical Asymptote: $x = 0$

   (b) Horizontal Asymptote:

$$\lim_{x\to\infty} 1 + \frac{1}{x} + \frac{1}{x^2} = 1$$

$$\lim_{x\to-\infty} 1 + \frac{1}{x} + \frac{1}{x^2} = 1$$

   So there is one horizontal asymptote at $y = 1$.

5. Find y'. Rewrite y as $y = 1 + x^{-1} + x^{-2}$.

$$y' = -x^{-2} - 2x^{-3}$$

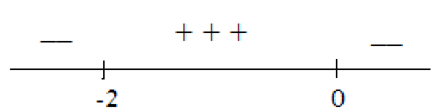
   To find critical values, we need to find out when $y' = 0$ or y' does not exist.

   We see y' does not exist when $x = 0$. Next, let's set $y' = 0$

$$\begin{aligned} -x^{-2} - 2x^{-3} &= 0 \\ -x^{-3}(x+2) &= 0 \end{aligned}$$

   So we have critical values at $x = 0$ and $x = -2$.

6. Use the number line to determine when y is increasing / decreasing.



Increasing: $(-2, 0)$

Decreasing: $(-\infty, -2)$ and $(0, \infty)$

Local Minimum: $(-2, \frac{3}{4})$

There is no local maximum. The number line indicates there would be one at $x = 0$. But recall that $x = 0$ is not in the domain.

7. Find y''

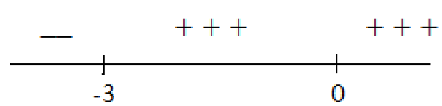
$$y'' = 2x^{-3} + 6x^{-4}$$

Critical values occur when $y'' = 0$ and y'' does not exist. You can see y'' does not exist when $x = 0$.

$$\begin{aligned} 2x^{-3} + 6x^{-4} &= 0 \\ 2x^{-4}(x + 3) &= 0 \end{aligned}$$

So we have critical values at $x = 0$ and $x = -3$

8. Use the number line to determine concavity.

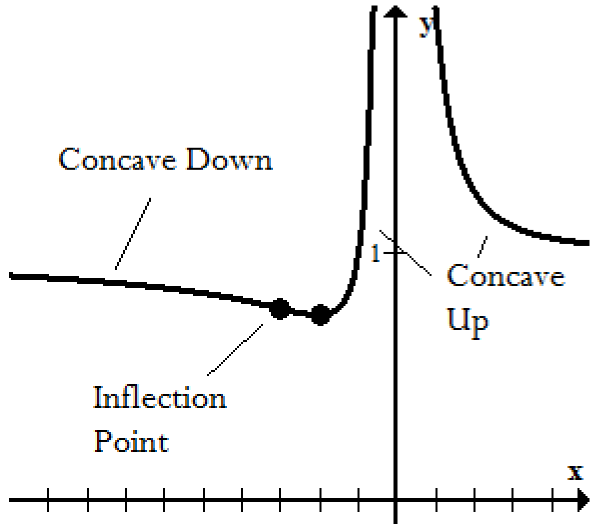


Concave Down: $(-\infty, -3)$

Concave Up: $(-3, 0)$ and $(0, \infty)$. You cannot write $(-3, \infty)$ because $x \neq 0$.

We have one point of inflection at $(-3, \frac{7}{9})$.

9. Time to sketch!


Concave Down
Concave Up
Inflection Point
y
x
1