2.7 Rates of Change

In this section we'll go over velocity and acceleration. We will also see how the second derivative can be useful to describe the behavior of velocity. For example, acceleration can tell us if the velocity is increasing or decreasing over time.

Definition 2.3 (Instantaneous Rate of Change)**.** If $s(t)$ is the position function that moves in a straight line (left or right), then $\lim_{\delta t \to 0} \dfrac{\delta s}{\delta t}$ represents the average velocity over δt (some change in time).

1. $v(t) = \dfrac{ds}{dt}$ is the instantaneous velocity and

$$v(t) = s'(t)$$

   Velocity is the derivative to the position function. This should make sense as a derivative is a rate of change and the change in position over change in time is velocity.

2. $a(t) = \dfrac{dv}{dt} = v'(t)$ is acceleration.

   Recall from previous math or physics courses that the change in velocity over change in time is acceleration. Acceleration tells us if an object is increasing its speed, decreasing its speed, or staying constant.

   For example, a car traveling at exactly 50 mph (never changing its speed) has an acceleration of 0.

The next example is very long. Are you ready? Here we go!

Example 2.39. The position of a particle is given by $s(t) = t^3 - 6t^2 + 9t$

1. Find the velocity at time t.

   When you're asked to find something at time t, it's just asking for that function. In this case, just find $v(t)$. Recall that $v(t)$ is $s'(t)$.

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

2. What is the velocity after 2 seconds?

   We have a velocity function. Just plug in $t = 2$ seconds into $v(t)$.

$$v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

   What does the negative sign mean?

   A negative sign on velocity indicates its movement (direction) is the opposite direction of what we consider forward. In this case, the negative sign means the particle is moving to the left (backwards).

3. What is the velocity after 4 seconds?

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

Consider these two velocities for a moment. At 2 seconds the particle is moving to the left at a speed of 3 m/s. At 4 seconds it's moving to the right at 9 m/s. That means at some point between 2 and 4 seconds, the particle had to stop and turn around.

4. When is the particle at rest?

As I mentioned in the last part, somewhere between 2 and 4 seconds, the particle had to stop and turn around. We want to know when the particle stops. Common sense says a particle is at rest when its velocity is 0 (i.e., not moving). So let's set $v(t) = 0$.

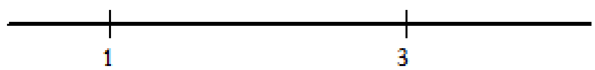
$$\begin{aligned} 3t^2 - 12t + 9 &= 0 \\ 3(t^2 - 4t + 3) &= 0 \\ 3(t-3)(t-1) &= 0 \end{aligned}$$

Solving for t, we get $t = 3$ and $t = 1$ seconds.

The particle is at rest after 1 seconds and 3 seconds. This makes sense that the particle stops at $t = 3$ seconds since we knew the particle had to stop somewhere between 2 and 4 seconds.

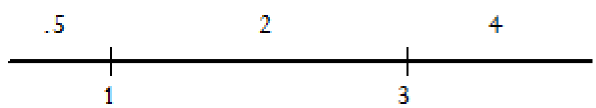
5. When is the particle moving forward?

A particle moves forward when its velocity is positive (+), and moving backwards when its velocity is negative (-). We are going to use a number line to figure this out. Here's what we're going to do. Plot on a number line when the velocity is 0 (when the particle is at rest).



When you evaluate $v(t)$, it will tell you its velocity. We are concerned if that velocity is positive (+) or negative (-). So here's what we're going to do,

(a) Pick a point in each region.



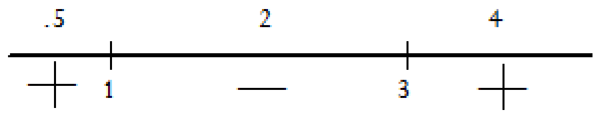
(b) Evaluate those points with $v(t)$ and determine if the velocity is (+) or (-).

$$v(.5) = (+)$$

$$v(2) = (-)$$

$$v(4) = (+)$$

(c) Mark this on the number line

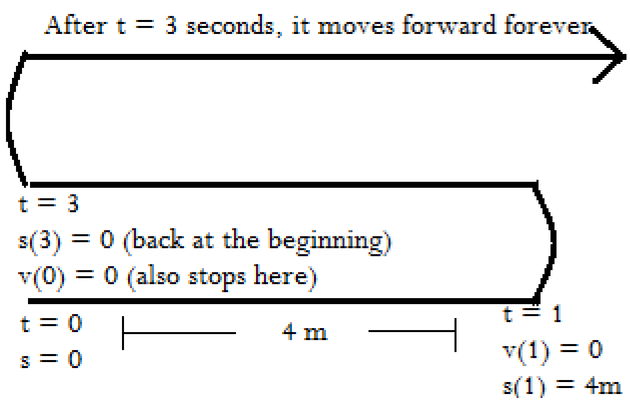


(d) So what does this tell us?

The particle is moving forward between 0 and 1 seconds and after 3 seconds.

The particle is moving backwards between 1 and 3 seconds.

6. Draw a diagram to represent the motion of the particle.



Note, the particle doesn't actually move up. I only draw it like that so we don't overlap the path of the particle.

7. Find the total distance traveled in the first 5 seconds.

This is a little trickier than just evaluating $s(5) = 20m$. What does $s(5)$ represent? It represents the distance away from the starting point (**displacement**). It doesn't represent an accumulation of the distance traveled over 5 seconds. The diagram is useful because it helps visually where the particle moved and how far it moved.

(a) At $t = 1$ the particle is 4m away from the starting point. So the accumulated distance traveled so far is 4m.

(b) At $t = 3$ seconds it's back at the starting point. We know this because $s(3) = 0$. This means it's traveled another 4m, for a total of 8m.

**Just for the sake of argument, suppose $s(3) = -1$ instead of $s(3) = 0$. This means at $t = 3$ seconds, the particle is 1m behind the starting point. So between $t = 1$ and $t = 3$ seconds, the particle actually moved 5m. But, $s(3) = 0$ so it only traveled 4m back.

(c) At $t = 5$ the particle is 20m away from the starting point. Since at $t = 3$ seconds it was at the starting point, it means between 3 and 5 seconds the particle traveled another 20m.

**Again for the sake or argument, suppose $s(3) = -1$. That would mean between 3 and 5 seconds the particle traveled $20 + 1 = 21$m since it was starting 1m behind the starting point.

(d) Anyhow, the accumulated distance traveled is $4 + 4 + 20 = 28m$.

8. Find the acceleration at time t.

Recall that $a(t) = v'(t)$.

$$a(t) = v'(t) = 6t - 12$$

9. Find the acceleration at $t = 1.5$ seconds, 2 seconds, 4 seconds.

   (a) $t = 1.5$ seconds

   $$a(1.5) = 6(1.5) - 12 = -3 \ \frac{m}{s^2}$$

   (b) $t = 2$ seconds

   $$a(2) = 6(2) - 12 = 0 \ \frac{m}{s^2}$$

   (c) $t = 4$ seconds

   $$a(4) = 6(4) - 12 = 12 \ \frac{m}{s^2}$$

10. When is the speeding up? When is it slowing down?

    Recall that acceleration tells us the rate of change in velocity. The following are the rules to determine if the particle is speeding up or slowing down.

(a) Speeding up: Both $v(t)$ and $a(t)$ must have the same sign.

$$v(t) > 0 \text{ and } a(t) > 0$$

$$v(t) < 0 \text{ and } a(t) < 0$$

(b) Slowing Down: $v(t)$ and $a(t)$ must have opposite signs.

$$v(t) > 0 \text{ and } a(t) < 0$$

$$v(t) < 0 \text{ and } a(t) > 0$$

(c) The way I think of it is $v(t)$ and $a(t)$ have to be working together to speed up (which is why they need to have the same sign). If they have opposite signs, it means acceleration is working against velocity (which is why it slows the particle down).

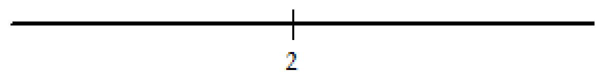
(d) So...let's find out when $a(t)$ is positive (+) and negative (-). We do this the same way we did it for $v(t)$. Start with a number line and plot the points when $a(t) = 0$.

First, we need to solve $a(t) = 6t - 12 = 0$.

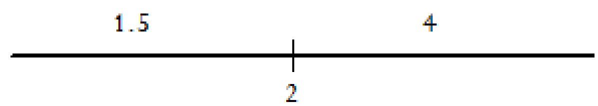
$$6t - 12 = 0$$

$$t = 2$$

Now plot $t = 2$ on the number line.



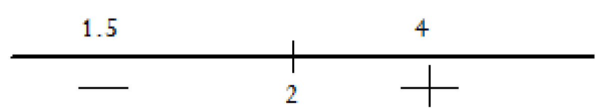
(e) Pick a point in each region and determine if acceleration is positive (+) or negative (-).



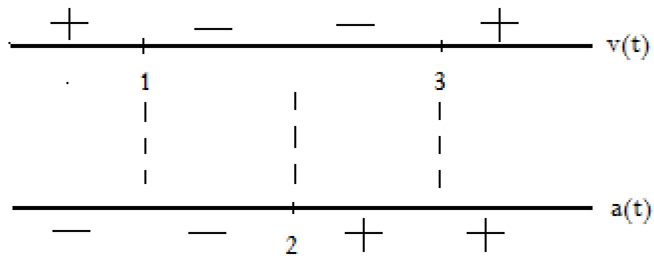
$$a(1.5) = -3(-)$$

$$a(4) = 12(+)$$

(f) Mark the regions with (+) or (-)



(g) Next, place the $v(t)$ number line above the $a(t)$ number line. Use this to see how the signs of $v(t)$ and $a(t)$ compare in different regions.



i. Between 0 and 1 seconds, $a(t)$ and $v(t)$ have opposite signs. So it's slowing down.

ii. Between 1 and 2 seconds, $a(t)$ and $v(t)$ have the same sign. So it's speeding up.

iii. Between 2 and 3 seconds, $a(t)$ and $v(t)$ have opposite signs. So it's slowing down.

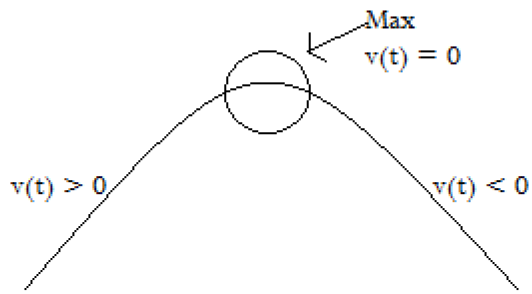
iv. After 3 seconds, $a(t)$ and $v(t)$ have the same sign. So it's speeding up.

Yay! We are finally done with that problem. I hope you now have a decent understanding of the relationship between the position function $s(t)$, the velocity function $v(t)$, and the acceleration function $a(t)$.

Example 2.40. If a ball is thrown vertically upward from the surface of the moon with a velocity 10 m/s, its height after t seconds is $h(t) = 10t - 0.83t^3$ (position function).

1. What is the max height reached?

   Take a look at the figure below.



The max height is reached when the ball stops moving up and begins to move down. In order to do that, it must stop moving. That's exactly when $v(t) = 0$.

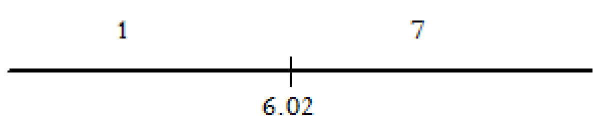
(a) Find $v(t)$

$$v(t) = h'(t) = 10 - 1.66t$$

(b) Set $v(t) = 0$

$$\begin{aligned} 10 - 1.66t &= 0 \\ -1.66t &= -10 \\ t &= \frac{10}{1.66} \\ t &= 6.02 \text{ seconds} \end{aligned}$$

(c) Let's use the number line to confirm the ball is moving up from 0 to 6.02 seconds and then down after 6.02 seconds. Pick a point from each region.



$$v(1) = 8.34\ (+)$$
$$v(7) = -1.62\ (-)$$

This confirms the ball is moving up from 0 to 6.02 seconds and down after 6.02 seconds.

(d) Plug $t = 6.02$ seconds into $h(t)$

$$h(6) = 10(6.02) - .83(6.02)^2 = 30.12 \text{ m}$$

The ball reaches a max height of 30.12 meters at 6.02 seconds.

2. What is the velocity when the ball is 25 meters above the ground?

(a) Find out when the ball is 25 m above the ground. Set $h(t) = 25$.

(b) Solve $h(t) = 25$

$$\begin{aligned} 10t - .83t^2 &= 25 \\ -.83t^2 + 10t - 25 &= 0 \end{aligned}$$

Solving this quadratic, we get $t = 3.54$ seconds and $t = 8.51$ seconds.

(c) Find the velocity at $t = 3.54$ and $t = 8.51$

$$v(3.54) = 10 - 1.66(3.54) = 4.1236 \text{ m/s}$$

$$v(8.51) = 10 - 1.66(8.51) = -4.1266 \text{ m/s}$$

3. What's $a(t)$?

$$a(t) = v'(t) = -1.66$$

So acceleration is constant. That makes sense because it's acceleration due to gravity (which is constant).

Also, notice that it's negative. Between 0 and 6.02 seconds, $v(t) > 0$. But $a(t) < 0$. Since they have opposite signs, the ball is slowing down. After 6.02 seconds, $v(t) < 0$. Now that $v(t)$ and $a(t)$ have the same sign, the ball is speeding up. That matches what we know happens on Earth. When we throw a ball up in the air, it slows down until it reaches its peak. After it reaches its peak, it starts to fall slowly. Over time it picks up speed.