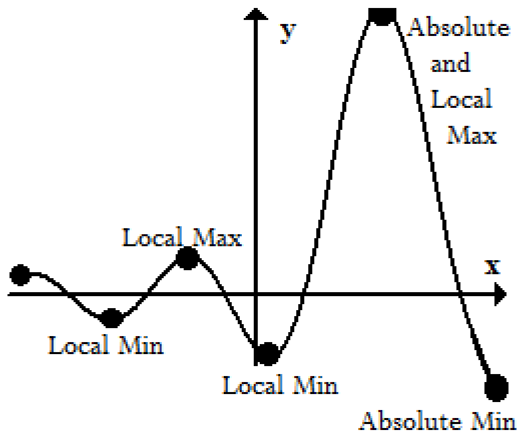
3 Application of Differentiation

3.1 Maximum and Minimum Values

Consider the following graph,



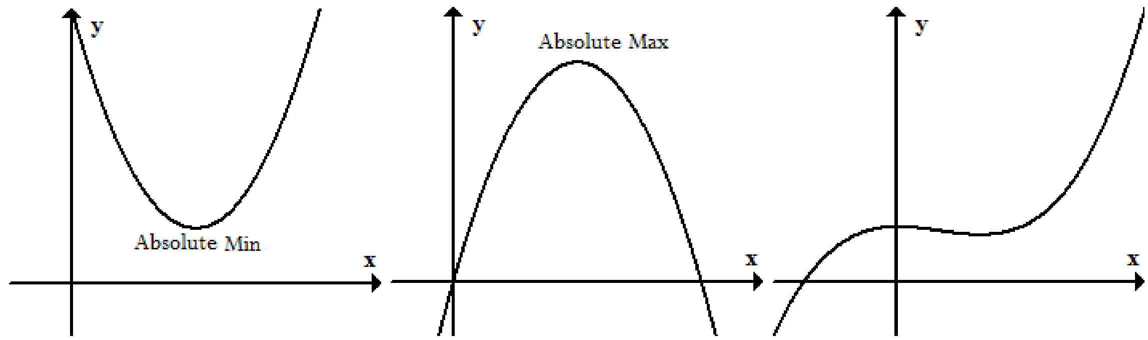
Definition 3.1 (Absolute Extrema)**.** A function f has an absolute max at $x = a$, if $f(a) \geq f(x)$ for all x in the domain. In other words, $f(a)$ has the largest y-value. $f(a)$ is the absolute max value.

A function f has an absolute min at $x = a$, if $f(a) \leq f(x)$ for all x in the domain. $f(a)$ is the absolute min value.

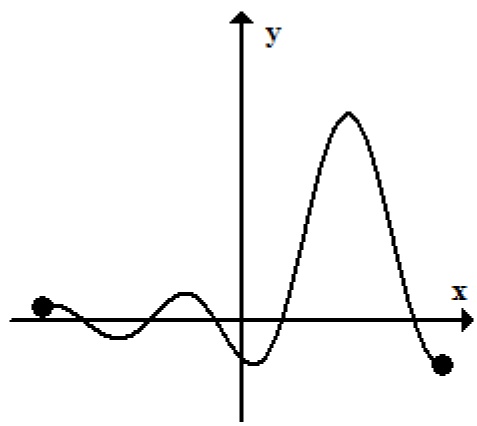
There can be **at most one** absolute max and **at most one** absolute min. From the graph you can see there can be many local max and local mins. Think of a local max as a peak. There can be many peaks on a graph. But there can be only be one LARGEST peak. Also note that an absolute max or absolute min can be an endpoint. We typically do not call endpoints local max or local mins even if they are the absolute max or absolute min.

All of these max and mins (local or absolute) are called **Extreme Values**.

Some functions can have an absolute max and no absolute min (or vice versa). There are functions that may not have any absolute max or absolute mins. The following are three functions that fit these scenarios.



Consider this graph again,

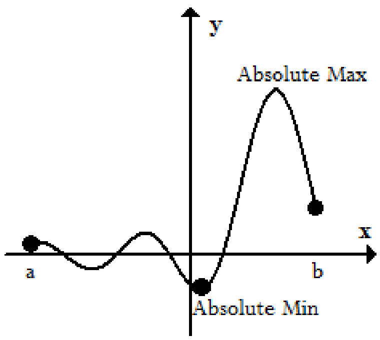


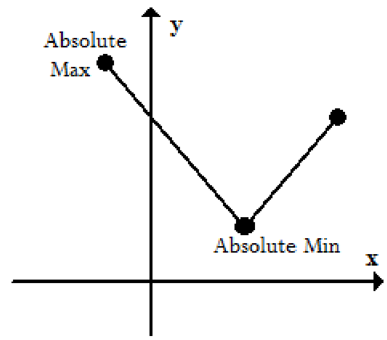
Why are we guaranteed an absolute max and absolute min in this graph? I haven't mentioned the theorem yet, but this graph satisfies certain requirements that guarantee an absolute max AND an absolute min.

3.1.1 The Extreme Value Theorem

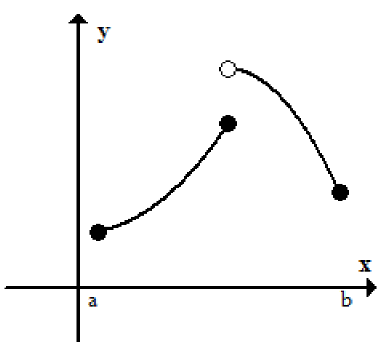
Theorem 3.1 (Extreme Value Theorem)**.** *If f is* ***continuous*** *on a* ***closed*** *interval [a,b], then f has an absolute max and absolute min somewhere in [a,b].*

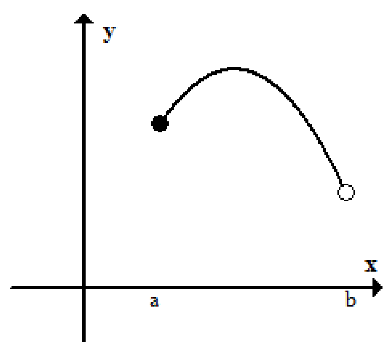
So that's it! As long as your function is continuous on a closed interval, it's guaranteed to have an absolute max and absolute min. Here are a couple of examples that satisfy the theorem's conditions.





So why are the two conditions in the theorem necessary? Take a look at the following two graphs.



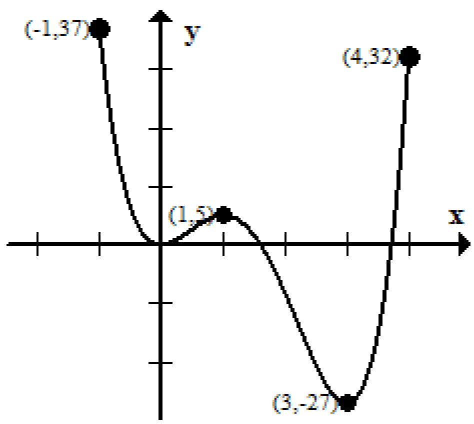


The first graph is on a closed interval [a,b] but is not continuous. You can see where the absolute max should have been. The discontinuity there messed it up. The second graph is continuous but not on a closed interval. We don't have an absolute min because it would have been at the endpoint b. Since its an open dot, we are never actually allowed to equal that y-value.

I want to make something very clear about theorems. If you satisfy the conditions of the theorem, you are guaranteed the conclusion. If f is continuous on a closed interval, we're guaranteed extreme values. Notice in the second graph, it does have an absolute max. The first graph does have an absolute min. Just because you don't satisfy the conditions, doesn't mean you can't have extreme values. Basically, if you don't satisfy the conditions of a theorem, then anything goes. Maybe you have extreme values, maybe you don't. Satisfy the conditions, and you're guaranteed extreme values.

Example 3.1. Let $f(x) = 3x^4 - 16x^3 + 18x^2$ on the interval $[-1, 4]$. Find the local max, local min, absolute max, and absolute min (if they exist). Another way of wording this question is, "Find all extrema."

Since we don't have a way to find local maximums and minimums yet, we'll just take a look at its graph.



1. Let's start with the absolute maximum.

   An absolute maximum is the largest (highest) y-value. Based on the graph, the absolute maximum occurs at the point (-1,37).

   (a) What is the absolute max? 37

(b) Where is the absolute max? $x = -1$

2. The absolute minimum occurs at the point (3,-27). Therefore, the absolute minimum is -27.

3. Local max?

   A local max is any point that is a peak. The graph must go down on both sides of the point. There is only one local max and occurs at the point (1,5).

4. Local min?

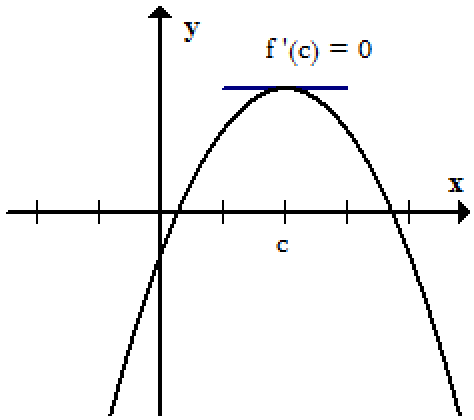
   A local min is the bottom of a dip. The graph must go up on both sides of the point. There are two local mins. One is at (0,0) and the other is at (3,-27).

5. I mentioned this already, but I don't allow endpoints to be local max or mins.

3.1.2 Fermat's Theorem

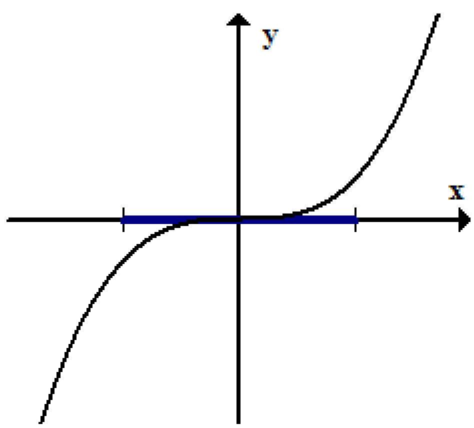
Theorem 3.2 (Fermat's Theorem)**.** *If f has a local minimum or local maximum at $x = c$, and if $f'(c)$ exists, then $f'(c) = 0$.*

What does this theorem really tell us? If $f'(c) = 0$, it's a **POTENTIAL** local max or local min. Take a look at the graph below.



It follows the theorem. We can see that it has a local max, and the derivative exists there. The conclusion from Fermat's Theorem guarantees us that the slope ($f'(c) = 0$).

As I stated already, if $f'(c) = 0$, it does not mean it's a local max or min, only that it could be. Consider the function $f(x) = x^3$.



Differentiate $f(x) = x^3$ and we get $f'(x) = 3x^2$. Plug in $x = 0$ and we get $f'(0) = 3(0)^2 = 0$. This shows we have a slope of 0 at $x = 0$; however, it is not a local max or local min. It is neither a peak nor the bottom of a dip.

If $f'(c) = 0$, it does not mean there is a local max or min.

If we want to find possible local max and local min locations, we find all places where $f'(x) = 0$ or where $f'(x)$ does not exist. They may not be local max or mins. But if we have them, they must be there.

3.1.3 Steps to find the absolute max or absolute min

1. Find all x-values where $f'(x) = 0$ or $f'(x)$ does not exist.

   These x-values are called **critical values**.

2. Find the y-value for each of the critical points in the interval. If no interval is given, assume $(-\infty, \infty)$.

3. Find the y-value for the endpoints. If the interval is open, we still find the y-value. If we find that the absolute max occurs at an endpoint that is open, then we conclude the function has no absolute max.

4. The largest y-value is the absolute max.

5. The smallest y-value is the absolute min.

Example 3.2. Find the absolute max and min (if they exist) of $f(x) = x^3 - 3x^2 + 1$ on the interval $\left[-\frac{1}{2}, 4\right]$.

Note, we have a continuous function on a closed interval. The Extreme Value Theorem concludes we **must** have an absolute max and absolute min.

1. Find the critical values.

   Set $f'(x) = 0$ and solve

$$f'(x) = 3x^2 - 6x$$

$$\begin{aligned} 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \end{aligned}$$

   Our critical values are $x = 0$ and $x = 2$. Since both are in the interval $[-1/2, 4]$, we use them both. Note, if any one of them wasn't in the interval, we don't use it.

2. Find the y-values for the critical values.

$$x = 0 :\rightarrow f(0) = 0^3 - 3(0)^2 + 1 = 1$$
$$x = 2 :\rightarrow f(2) = 2^3 - 3(2)^2 + 1 = -3$$

3. Find the y-values at the endpoints.

$$x = -\frac{1}{2} :\rightarrow f(-1/2) = \frac{1}{8}$$
$$x = 4 :\rightarrow f(4) = 17$$

4. Find the absolute max and absolute min.

   (a) Absolute max: (4,17)

   (b) Absolute min: $(2, -3)$

3.1.4 Steps to find the local maximums and minimums

This material shows up again in section 3, but we might as well cover it now. I've already stated that to find a local max or min, we first find all x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Let's continue with the previous example.

1. Find the critical values.

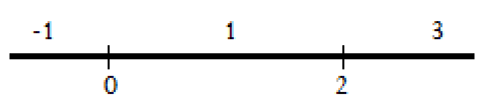
   We already solved $f'(x) = 0$ and found critical values at $x = 0$ and $x = 2$. Since $f(x)$ a polynomial, the derivative will exist everywhere. So we don't have to worry about $f'(x)$ not existing.

2. Plot the critical values on a number line.

3. Pick a number from each region and plug it into $f'(x)$. We just need to know if $f'(x)$ is positive (+) or negative (-).

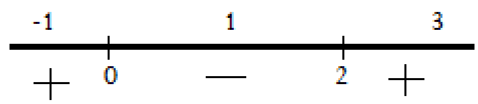
   Remember that $f'(x)$ tells us the slope of the graph. If the slope is positive, the function is going up. If the slope is negative, the function is going down.

$$f'(-1) = 3(-1)(-1-2) = 9 \ (+)$$

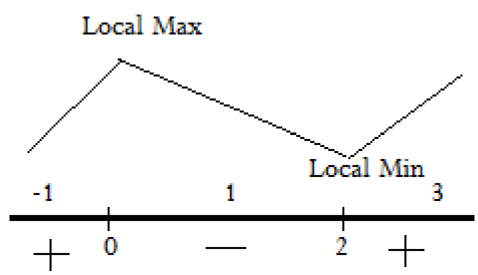
$$f'(1) = 3(1)(1-2) = -3 \ (-)$$

$$f'(3) = 3(3)(3-2) = 9 \ (+)$$



This means the graph has to have the following form. It has to be increasing to $x = 0$ and then start decreasing. That makes a peak (local max) at $x = 0$.

It decreases to $x = 2$ and then begins to increase. That makes a dip (local min) at $x = 2$.



4. Find the local max.

Now that we know the local max occurs at $x = 0$, we need to find its y-value. DO NOT PLUG $x = 0$ into $f'(x)$. To find the y-value, we need to plug it in $f(x)$.

$$f(0) = 1$$

Local Max: (0,1)

5. Find the local min.

   Now that we know the local min occurs at $x = 2$, we need to find its y-value. To find the y-value, we need to plug it in $f(x)$.

$$f(2) = -3$$

   Local Max: (2,-3)

Example 3.3. Let's jump into a harder example. It's harder because solving $f'(x) = 0$ is difficult. Anyhow, let's get started. Find the local extrema for $f(x) = x^{4/5}(x-4)^2$.

1. Find $f'(x)$. Use the product rule.

$$\begin{aligned} f'(x) &= x^{4/5} \cdot \frac{d}{dx}\left[(x-2)^2\right] + (x-4)^2 \cdot \frac{d}{dx}\left[x^{4/5}\right] \\ &= x^{4/5} \cdot [2(x-4) \cdot 1] + (x-4)^2 \cdot \frac{4}{5}x^{-1/5} \end{aligned}$$

   Since we have to set $f'(x) = 0$, we need to simplify $f'(x)$.

$$\begin{aligned} x^{4/5} \cdot [2(x-4) \cdot 1] + (x-4)^2 \cdot \frac{4}{5}x^{-1/5} &= 0 \\ 2x^{-1/5}(x-4)\left[x + \frac{2}{5}(x-4)\right] \&= & 0 \\ 2x^{-1/5}(x-4)\left[\frac{7}{5}x - \frac{8}{5}\right] \end{aligned}$$

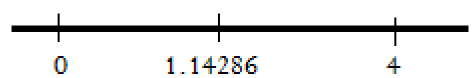
   There are three factors here.

   (a) $x^{-1/5}$: This means $f'(x)$ does not exist at $x = 0$ since $0^{-1/5}$ does not exist.

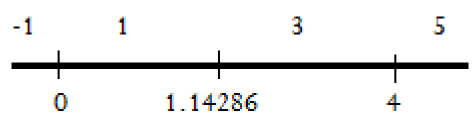
(b) $(x-4)$: Setting this equal to 0, we get $x=4$.

(c) $\left[\frac{7}{5}x-\frac{8}{5}\right]$: Setting this equal to 0, and we get $x=\frac{8}{7}$ or $x\approx 1.14286$

2. Plot the critical values on the number line.



3. Pick a number from each region and plug it in $f'(x)$.



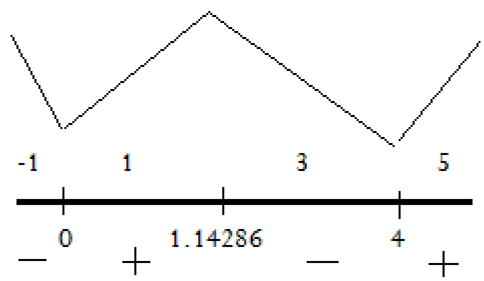
$$x=-1;\rightarrow f'(-1)=(-1)^{-1/5}(2.8(-1)^2-14.4(-1)+12.8)=-30$$

$$x=1;\rightarrow f'(1)=(1)^{-1/5}(2.8(1)^2-14.4(1)+12.8)=1.2$$

$$x=3;\rightarrow f'(3)=(3)^{-1/5}(2.8(3)^2-14.4(3)+12.8)=-4.174256$$

$$x=5;\rightarrow f'(5)=(5)^{-1/5}(2.8(5)^2-14.4(5)+12.8)=7.82762$$

Our number line now looks like this,



4. Find the local maximums.

   There is a local max at $x = 1.14286$. Plug $x = 1.14286$ into $f(x)$ to find the y-value.

   Local Max: (1.14286, 9.083588)

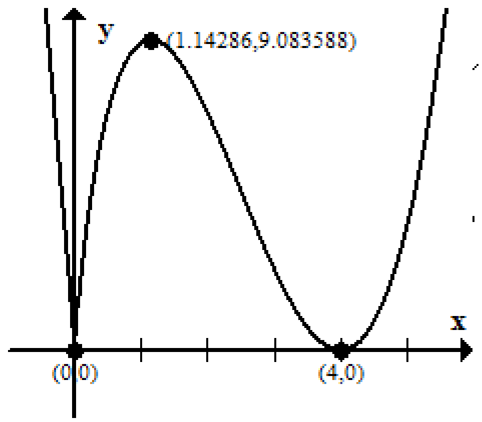
5. Find the local minimums.

   There are local minimums at $x = 0$ and $x = 4$.

   Warning: The critical value $x = 0$ came from when $f'(0)$ did not exist. That means it's possible $f(0)$ may not exist. If $f(0)$ does not exist, then its not a local min. If it does exist, it is a local min. Fortunately for us, $f(0)$ does exist.

   Local Mins: (0, 0) and (4,0)

Let's go ahead and look at the graph of $f(x) = x^{4/5}(x-4)^2$ to verify our findings.



Sweet! Also, do you see how the graph makes a corner at (0,0). That's why $f'(0)$ did not exist. Hopefully you can recall the different scenarios why a derivative may not exist.