## 3.4 Limits at Infinity - Asymptotes

**Definition 3.3.** If $f$ is a function defined on some interval $(a, \infty)$, then $\lim_{x\to\infty} f(x) = L$ means that values of $f(x)$ are very close to $L$ (keep getting closer to $L$) as $x \to \infty$.

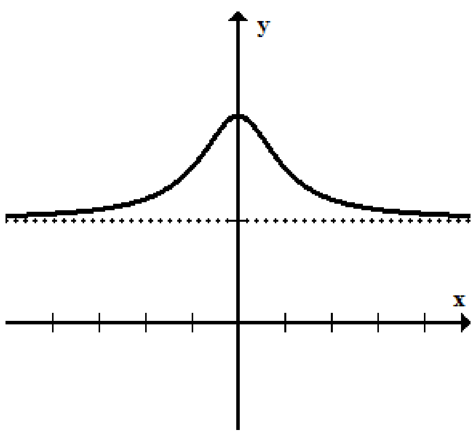
The line $y = L$ is called a horizontal asymptote of $f(x)$ if either

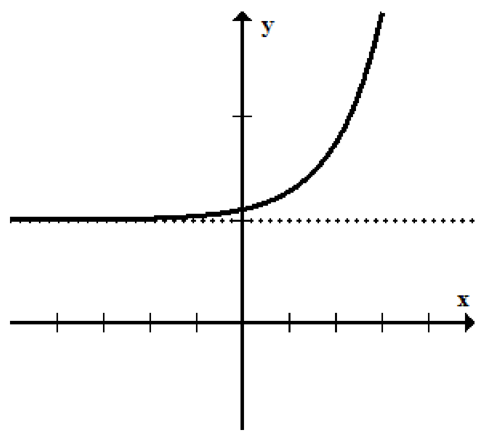
$$\lim_{x\to\infty} f(x) = L$$

or

$$\lim_{x\to-\infty} f(x) = L$$

Here's an example of a function with ONE asymptote

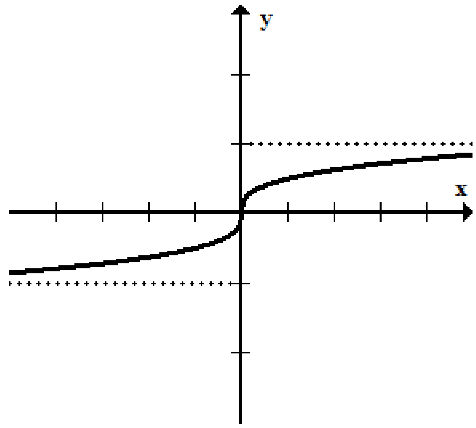





Notice that both graphs have a horizontal asymptote at $y = 1$. The first graph has the horizontal asymptote as $x \to -\infty$ and $x \to \infty$. The second graph only has the horizontal asymptote as $x \to -\infty$. So what does this mean?

When you determine an asymptote, don't always assume it's on both sides of the graph.

Here's an example of a graph that has two horizontal asymptotes.




Do you see how this graph has two horizontal asymptotes? One is $y = -1$ as $x \to -\infty$ and the other is $y = 1$ as $x \to \infty$.

At this point, you probably know about **vertical asymptotes**, but I want to go over them briefly.

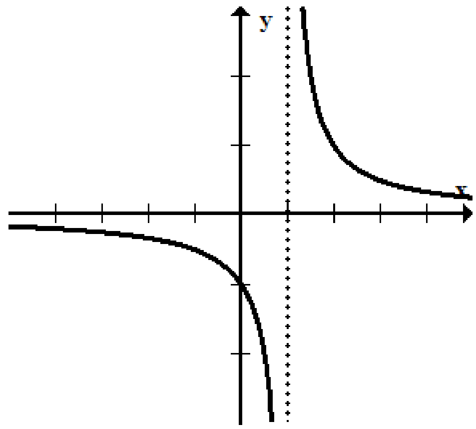
If $\lim_{x \to a} f(x) = \pm\infty$, then $x = a$ is a **vertical asymptote**.

One way to find them is to set the denominator equal to 0. If $x = a$ makes the denominator 0 but NOT the numerator, then $x = a$ is a vertical asymptote.

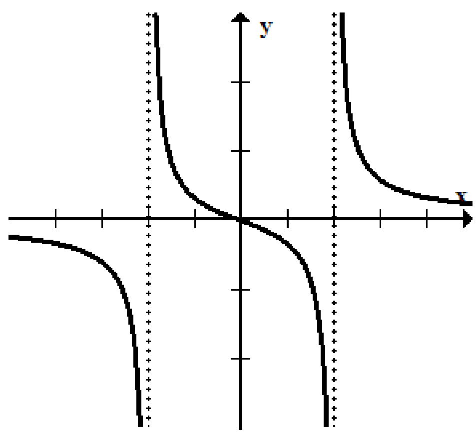
**Warning!** If $x = a$ makes the numerator and denominator both 0, it means it may or may not be a vertical asymptote. You have to follow through with the limit to confirm.

**Example 3.15.**

1. $f(x) = \dfrac{1}{x-1}$ has a vertical asymptote at $x = 1$



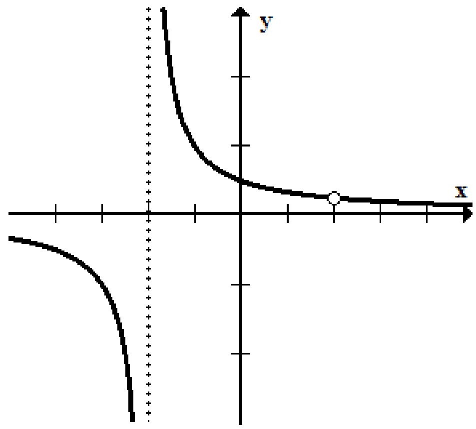

2. $f(x) = \dfrac{x}{x^2 - 4}$ has vertical asymptotes at $x = -2$ and $x = 2$




3. $f(x) = \dfrac{x-2}{x^2 - 4}$ has a vertical asymptote at $x = -2$ but not at $x = 2$. Why is that? Even though the denominator is 0 when you plug in $x = 2$, look at the limit

$$\begin{aligned}
\lim_{x \to 2} \frac{x-2}{x^2-4} &= \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} \\
&= \lim_{x \to 2} \frac{1}{x+2} \\
&= \frac{1}{4} \\
&\neq \infty
\end{aligned}$$

You can see that we do NOT have a vertical asymptote at $x = 2$.




The shortcut methods to finding vertical and horizontal asymptotes can be found in 'Types of Functions'

Before moving on to sketching graphs with asymptotes, I want to do some examples of finding them without the shortcuts. They involve a bit of algebra.

**Example 3.16.**

1. Find the horizontal asymptote(s) of $f(x) = \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

   We need to find two different limits.

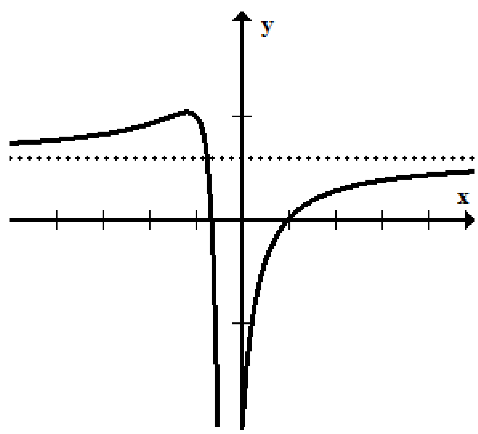
   (a) As $x \to \infty$

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\ &= \frac{3 - 0 - 0}{5 + 0 + 0} \\ &= \frac{3}{5} \end{aligned}$$

(b) As $x \to -\infty$

$$\begin{aligned}\lim_{x\to-\infty}\frac{3x^2-x-2}{5x^2+4x+1} &= \lim_{x\to-\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}\\ &= \frac{3-0-0}{5+0+0}\\ &= \frac{3}{5}\end{aligned}$$

So there is only one horizontal asymptote, $y = \dfrac{3}{5}$




2. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

(a) Let's start with the vertical asymptote. What makes the denominator $3x-5=0$?

$$x = \frac{5}{3}$$

Since $x = \dfrac{5}{3}$ does not make the numerator 0, we know $x = \dfrac{5}{3}$ is a vertical asymptote.

(b) Now on to the horizontal asymptotes. This one is going to be tricky.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2x^2\left(1+\dfrac{1}{2x^2}\right)}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2}|x|\sqrt{1+\dfrac{1}{2x^2}}}{3x-5}$$

Recall: $|x| = x$ when $x > 0$

$$= \lim_{x\to\infty} \frac{\sqrt{2}x\sqrt{1+\dfrac{1}{2x^2}}}{3x-5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2}\sqrt{1+\dfrac{1}{2x^2}}}{3-\dfrac{5}{x}}$$

$$= \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{2x^2\left(1+\frac{1}{2x^2}\right)}}{3x-5}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{2}|x|\sqrt{1+\frac{1}{2x^2}}}{3x-5}$$

Recall: $|x| = -x$ when $x < 0$

$$= \lim_{x\to-\infty} \frac{\sqrt{2}(-x)\sqrt{1+\frac{1}{2x^2}}}{3x-5}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{2}\sqrt{1+\frac{1}{2x^2}}}{3-\frac{5}{x}}$$

$$= \frac{-\sqrt{2}}{3}$$

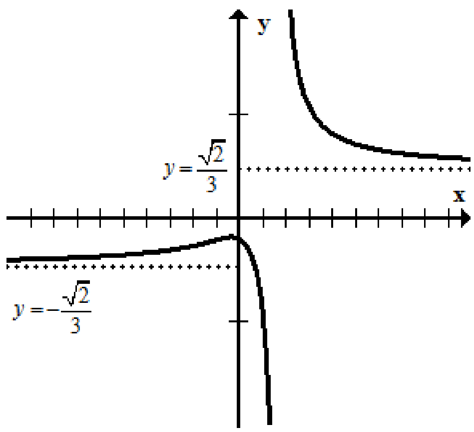
So we have two horizontal asymptotes. Now this seemed a bit algebraic intensive. Any way to make it easier? The answer is yes, but make sure you can do the above method.

**Shortcut**

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2x^2}}{3x} = \lim_{x\to\infty} \frac{\sqrt{2}x}{3x} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{2x^2}}{3x} = \lim_{x\to-\infty} \frac{\sqrt{2}(-x)}{3x} = \frac{-\sqrt{2}}{3}$$

Here's the graph of $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$




3. Evaluate the limit $\lim_{x\to\infty} \sqrt{x^2+6x} - x$.

If you try to "evaluate" this, you get $\infty - \infty$. This does not necessarily mean 0. Let's go ahead and find out.

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x^2+5x}-x &= \lim_{x\to\infty} \left(\sqrt{x^2+5x}-x\right)\cdot \frac{\sqrt{x^2+5x}+x}{\sqrt{x^2+5x}+x} \\
&= \lim_{x\to\infty} \frac{x^2+6x-x^2}{\sqrt{x^2+5x}+x} \\
&= \lim_{x\to\infty} \frac{6x}{\sqrt{x^2+5x}+x} \\
&= \lim_{x\to\infty} \frac{6x}{\sqrt{x^2(1+\frac{5}{x})}+x} \\
&= \lim_{x\to\infty} \frac{6x}{x\left(\sqrt{1+\frac{5}{x}}+1\right)} \\
&= \lim_{x\to\infty} \frac{6}{\sqrt{1+\frac{5}{x}}+1} \\
&= \frac{6}{2} \\
&= 3
\end{aligned}$$

4. Evaluate $\lim_{x\to\infty} x - x^2$

The above method doesn't really work. Instead, note that we turn this into a product of two functions by factoring out $x$.

$$\lim_{x\to\infty} x(1-x) = \infty \cdot -\infty = -\infty$$

Keep in mind that my notation isn't formal. But it should help you understand what just happened. It's a product of two functions. So I get a very large number times a very large negative number. Common sense says this should equal an even larger negative number.

### 3.4.1 Examples of Curve Sketching with Asymptotes

Refer to the section **Summary of Curve Sketching** for an outline and guide to help you with curve sketching.

Sketch $y = \dfrac{2x^2}{x^2-1}$

1. Domain: $x \neq \pm 1$

2. Intercepts:

$$x-\text{intercept: } (0,0)$$
$$y-\text{intercept: } (0,0)$$

3. Asymptotes:

   (a) Vertical Asymptote: $x = -1$ and $x = 1$

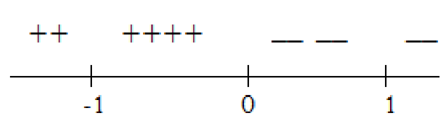
   (b) Horizontal Asymptote: $y = 2$ because $\displaystyle\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = 2$

4. Find $y'$

$$\begin{aligned} y' &= \frac{(x^2-1)\cdot(4x) - 2x^2\cdot(2x)}{(x^2-1)^2} \\ &= \frac{-4x}{(x^2-1)^2} \end{aligned}$$

   We have critical values at $x=0$, $x=-1$ and $x=1$.

5. Use the number line to determine when $y$ is increasing or decreasing.



Increasing: $(-\infty, -1)$ and $(-1, 0)$. You can't write $(-\infty, 0)$ because $x = -1$ is not in the domain.

Decreasing: $(0, 1)$ and $(1, \infty)$. You can't write $(0, \infty)$ because $x = 1$ is not in the domain.

We have a local max at $x = 0$. As a point, it's $(0, 0)$.

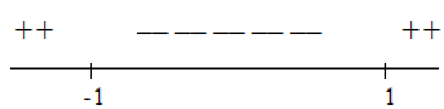
6. Find $y''$

   Rewrite $f'(x)$ as $f'(x) = -4x(x^2 - 1)^{-2}$ so you can use the product rule.

$$\begin{aligned} f''(x) &= -4x \cdot -2(x^2-1)^{-3} \cdot 2x + (x^2-1)^{-2} \cdot (-4) \\ &= -4(x^2-1)^{-3}\left[-4x^2 + (x^2-1)\right] \\ &= -4(x^2-1)^{-3}\left[-3x^2 - 1\right] \\ &= \frac{4(3x^2+1)}{(x^2-1)^3} \end{aligned}$$

   We have critical values at $x = -1$ and $x = 1$. Notice that we get no critical values from the numerator since $3x^2 + 1 \neq 0$.

7. Use the number line to determine when $y$ is concave up or down.

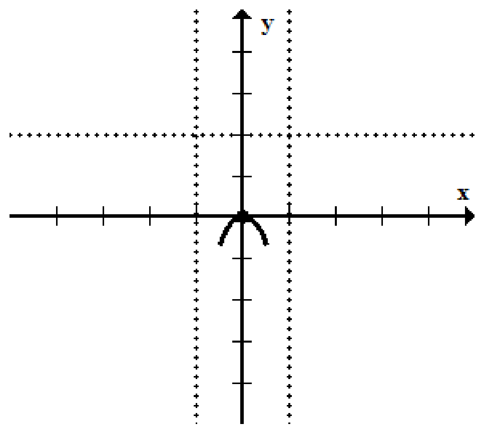



   Concave Up: $(-\infty, -1)$ and $(1, \infty)$

   Concave Down: $(-1, 1)$

We have a change of concavity at $x = -1$ and $x = 1$, but these are asymptotes. So they cannot be **points** of inflection.

8. Now, let's piece everything together to sketch the graph. First, let's start with our intercepts, important points like local extreme and points of inflection, and the asymptotes.




9. Now finish the graph using your knowledge of when the function is increasing / decreasing and its concavity.