4.5 Integration by Substitution

Since the fundamental theorem makes it clear that we need to be able to evaluate integrals to do anything of decency in a calculus class, we encounter a bit of a problem when we have an integral like

$$\int (2x+1)\cos(x^2+x)\;dx.$$

We cannot compute this integral, since the integrand is a product, and we have no integration rule which tells us how to deal with a product. This is the first section we have which covers a special technique of integration, one that helps us to reconcile integrals such as the above one. This first technique involves the introduction of a new variable – here we let the new variable, u, equal the value in the cosine function, x^2+x. Then, the differential of u is $du = (2x+1)\;dx$, so doing a substitution of $u = x^2+x$ and $dx = \dfrac{du}{2x+1}$, we have

$$\begin{aligned}\int (2x+1)\cos(x^2+x)\;dx &= \int (2x+1)\cos(u)\;\frac{du}{2x+1}\\ &= \int \cos(u)\;du\\ &= \sin(u)+C = \sin(x^2+x)+C.\end{aligned}$$

But is this right? There is only one way to check, by differentiating:

$$\frac{d}{dx}\sin(x^2+x)+C = \cos(x^2+x)\cdot\frac{d}{dx}\left[x^2+x\right] = \cos(x^2+x)(2x+1)$$

So, yes this works. As a matter of fact, this will always work for every integral which has the form

$$\int f(g(x))g'(x)dx.$$

After all, if $F' = f$, then

$$\int F'(g(x))g'(x)dx = F(g(x)) + C,$$

because, by the Chain Rule, we have

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot \frac{d}{dx}g(x) = F'(g(x)) \cdot g'(x).$$

So, what seems more clear now, is that we have to make a "change of variable," a substitution from the variable x to a variable u, where $u = g(x)$. Then, we have

$$\int F'(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int F'(u)du,$$

by writing $F' = f$, we have

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

This proves the following integration technique:

4.5.1 Integration by Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Note that we had to use the Chain Rule to prove this – meaning that once we define u, we need to use the chain rule to find du as well, which will have a dx in it. We do allow algebra with these differentials in order to solve for dx, which will help in the substitution process.

Example 4.34. Find

$$\int 5x^3 \sec^2(x^4 - 7)dx.$$

A common guideline for this u-substitution is to let u equal the inside of the most complicated function. After all, functions don't often get easier by differentiating! Since the $\sec^2()$ is more complicated and uglier than a cubic, we let $u = x^4 - 7$. Then, $du = 4x^3 dx$. We solve for dx:

$$dx = \frac{du}{4x^3}$$

and we now make our substitutions to let us integrate:

$$\begin{aligned}\int 5x^3 \sec^2(x^4 - 7)dx &= \int 5x^3 \sec^2(u) \cdot \frac{du}{4x^3} \\ &= \frac{5}{4} \int \sec^2(u)du \\ &= \frac{5}{4} \tan(u) + C \\ &= \frac{5}{4} \tan\left(x^4 + 7\right) + C\end{aligned}$$

Note that at the end of this problem, we MUST put the integral back in terms of its original variable, x.

The main idea behind the u-substitution rule is the replace a complicated function with something much easier to work with, which we do by replacing a function of x by a variable u. The main difficulty here is to determine the function for which we are going to substitute. If you can identify part of the integrand as the derivative of another part, that will be the du piece and the other the u piece. However, if you cannot, a common theme is to let a complicated part of the integrand be u. There is no exact method to find a u that works, its more like finding how much powdered sugar goes in cake icing – we start simply by guessing.

Example 4.35. Evaluate

$$\int 4x\sqrt{8x^2+11}dx.$$

Let $u = 8x^2 + 11$, so that $du = 16x\ dx$, and we have $dx = \dfrac{du}{16x}$. Then by u-substitution we have

$$\begin{aligned}\int 4x\sqrt{8x^2+11}dx &= \int 4x\sqrt{u}\frac{du}{16x}\\ &= \frac{1}{4}\int u^{1/2}du\\ &= \frac{1}{4}\cdot\frac{1}{3/2}u^{3/2}+C\\ &= \frac{1}{6}\left(8x^2+11\right)^{3/2}+C\end{aligned}$$

If we tried the other function, and we let $u = 4x$, then $du = 4dx$, and there is nothing we can do about the square root. Hence, we work with the more complicated function.

Example 4.36. Find

$$\int \frac{(x+3x^2)dx}{\sqrt[3]{x^2+2x^3}}.$$

We have two functions here, a numerator of $x + 3x^2$ and a denominator of $\sqrt[3]{x^2+2x^3}$. Clearly, the denominator is more complicated, so we start there: let $u = x^2 + 2x^3$. Then $du = 2x + 6x^2$, so

$$dx = \frac{du}{2x+6x^2},$$

and we have

$$\begin{aligned}
\int \frac{(x+3x^2)dx}{\sqrt[3]{x^2+2x^3}} &= \int \frac{x+3x^2}{\sqrt[3]{u}} \frac{du}{2x+6x^2} \\
&= \int (x+3x^2) \cdot u^{-1/3} \cdot \frac{du}{2(x+3x^2)} \\
&= \int \frac{1}{2} u^{-1/3} du \\
&= \frac{1}{2} \frac{1}{2/3} u^{2/3} + C \\
&= \frac{3}{4} u^{2/3} + C
\end{aligned}$$

Example 4.37. Find

$$\int \sqrt{3x+5}dx.$$

There is really only one option here for u, the exponent: $u = 3x+5$, so $du = 3dx$ and $dx = \frac{du}{3}$, and we have

$$\int \sqrt{3x+5}dx = \int \sqrt{u}\, \frac{du}{3} = \frac{1}{3}\int \sqrt{u}\, du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + C = \frac{2}{9}(3x+5)^{3/2} + C.$$

If you get to the point where you can evaluate integrals via substitution without directly writing down the substitution, that is fine. However, it can be dangerous to do so, in that you can lose track of constants quite easily. Here, we will constantly, explicitly, give the exact substitution and do the cancellation, just to make sure we all understand what's going on. Don't let ego get in the way. Taking an extra minute or two to write down the explicit substitution and do the cancellation is much better than wasting 10 minutes trying to find where the constant error came from.

Example 4.38. Find

$$\int x^5 \sqrt{1-x^3}dx.$$

Now, this is interesting. Even if we let $u = 1 - x^3$, we won't be able to clear out that x^5 on the outside. But letting $u = x^4$ won't work either, as we won't be able to clear out that pesky root. We note that we can write $x^5 = x^2 \cdot x^3$, and we do see some of du there. Thus, we let $u = 1 - x^3$ and $du = -3x^2 dx$. This gives

$$\begin{aligned}
\int x^5\sqrt{1-x^3}dx =& \int x^3 \cdot x^2 \sqrt{u}\frac{du}{-3x^2} \\
=& \frac{-1}{3}\int x^3\sqrt{u}du \\
& \text{but we can write } x^3 = 1 - u, \text{ so} \\
=& \frac{-1}{3}\int (1-u)\cdot u^{1/2}du \\
=& \frac{-1}{3}\int u^{1/2} - u^{3/2}du \\
=& \frac{-1}{3}\left(\frac{1}{3/2}u^{3/2} - \frac{1}{5/2}u^{5/2}\right) + C \\
=& \frac{-1}{3}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \\
=& \frac{2}{15}u^{5/2} - \frac{2}{9}u^{3/2} + C \\
=& \frac{2}{15}(1-x^3)^{5/2} - \frac{2}{9}(1-x^3)^{3/2} + C
\end{aligned}$$

Example 4.39. Find

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$$

Let $u = \sqrt{x}$ and $du = \dfrac{1}{2\sqrt{x}}\, dx$. This gives us $dx = 2\sqrt{x}\, du$. Do not substitute both $\sqrt{x}$s as u. Only the one in the sine function becomes u.

$$\begin{aligned}
\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx &= \int \frac{\sin(u)}{\sqrt{x}}\,(2\sqrt{x}du) \\
&= \int 2\sin(u) \\
&= -\cos(u) + C \\
&= -\cos(\sqrt{x}) + C
\end{aligned}$$

Even though the substitution looked a bit tricky, it turned out ok.

Example 4.40. Find

$$\int \frac{\csc^2(x)}{\cot^4(x)}\,dx$$

We let $u = \cot(x)$ because we know $\dfrac{d}{dx}\cot(x) = \csc^2(x)$ which is in the integral. You just have to practice these in order to get better at identifying u. Then $du = -\csc^2(x)\,dx$, which gives us $dx = \dfrac{du}{-\csc^2(x)}$.

$$\begin{aligned}
\int \frac{\csc^2(x)}{\cot^4(x)}\,dx &= \int \frac{\csc^2(x)}{u^4}\,\frac{du}{-\csc^2(x)} \\
&= \int -\frac{1}{u^4}\,du \\
&= \int -u^{-4}\,du \\
&= -\frac{u^{-3}}{-3} + C \\
&= \frac{1}{3}(\cot(x))^{-3} + C
\end{aligned}$$

4.5.2 Definite Integrals

We have two different methods for evaluating definite integrals. The first is quite similar to what we have been doing with the indefinite integrals – we perform our substitution, integrate and revert back to our original variable before we evaluate at the integration bounds. For example,

$$\begin{aligned}\int_0^{\pi} \sin(3x-1)dx =& \int_{x=0}^{x=\pi} \sin(u)\frac{du}{3}\\ =&\frac{1}{3}\int_{x=0}^{x=\pi} \sin(u)du\\ =&\frac{-1}{3}\cos(u)\mid_{x=0}^{x=\pi}\\ =&\frac{-1}{3}\cos(3x-1)\mid_0^{\pi}\\ =&\frac{-1}{3}\left(\cos(3\pi-1)-\cos(1)\right)\end{aligned}$$

Which is all lovely, in that it works. However, there is a second way, which is occasionally easier – we change the bounds on integration when we change the variable:

4.5.3 u-Substitution for Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range $u=g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

We give a proof, and then demonstrate on our previous function.

Proof. Let F be a function such that $F'=f$. Then, by the Chain Rule, $F(g(x))$ is an anti-derivative of $f(g(x))g'(x)$. We use Part 2 of the FTC to get

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\mid_a^b = F(g(b))-F(g(a)).$$

However, we can apply the FTC a second time, and we get

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\mid_{g(a)}^{g(b)} = F(g(b))-F(g(a)).$$

Since they are identical, the result is proves.

□

Example 4.41. Using the second method for u-substitution, find

$$\int_0^\pi \sin(3x-1)dx.$$

We perform the same u-sub, with $u = 3x - 1$. Then, $u(0) = -1$ and $u(\pi) = 3\pi - 1$. This gives

$$\begin{aligned}\int_0^\pi \sin(3x-1)dx &= \int_{u=-1}^{u=3\pi-1} \sin(u)\frac{du}{3} \\ &= \frac{-1}{3}\cos(u)\Big|_{-1}^{3\pi-1} \\ &= \frac{-1}{3}\left(\cos(3\pi-1)-\cos(-1)\right)\end{aligned}$$

Note that we did NOT have to return to the original variable after integrating, which makes the problem a little easier. The method you use is entirely up to you – there will be no assignment which forces you into one over the other. This second method is a little faster, but also a little more apt to minor error.

Example 4.42. Evaluate

$$\int_0^3 \frac{7x}{(1+7x^2)^2}dx.$$

We let $u = 1 + 7x^2$ so that $du = 14xdx$. We also have $u(0) = 1$ and $u(3) = 1 + 63 = 64$, so we have

$$\begin{aligned}\int_0^3 \frac{7x}{(1+7x^2)^2}dx &= \int_1^{64} \frac{7x}{u^2}\frac{du}{14x} \\ &= \frac{1}{2}\int_1^{64} u^{-2}du \\ &= \frac{1}{2}\cdot\left(\frac{1}{-1}u^{-1}\Big|_1^{64}\right) \\ &= \frac{1}{2}\left(-(64)^{-1}+(1)^{-1}\right) \\ &= \frac{1}{2}\left(1-\frac{1}{64}\right) = \frac{63}{128}\end{aligned}$$

Example 4.43. Find

$$\int_0^{\pi/4} \tan^3(x)\sec^2(x)\;dx$$

Let $u = \tan(x)$. We do this because $\dfrac{d}{dx}\tan(x) = \sec^2(x)$, which is in the integral. So $du = \sec^2(x)\;dx$, which gives us $dx = \dfrac{du}{\sec^2(x)}$.

We now change the bounds. If $x = 0$, then $u(0) = \tan(0) = 0$. If $x = \pi/4$, then $u(\pi/4) = \tan(\pi/4) = 1$. Thus,

$$\int_0^{\pi/4} \tan^3(x)\sec^2(x)\;dx = \int_0^1 u^3\sec^2(x)\;\frac{du}{\sec^2(x)} = \int_0^1 u^3\;du = \left.\frac{u^4}{4}\right|_0^1 = \frac{1}{4}$$

We give one final theorem for this chapter, which helps simplify certain integrals if the function satisfies a nice condition:

Theorem 4.2. *Suppose that f is continuous on $[-a,a]$.*

1. *If f is even (meaning that $f(-x) = f(x)$), then*
$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$
2. *If f is odd (meaning that $f(-x) = -f(x)$), then*
$$\int_{-a}^{a} f(x)dx = 0.$$

Proof. We begin by splitting the integral into two pieces

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx = -\int_0^{-a} f(x)dx + \int_0^a f(x)dx.$$

In the first integral, we make the substitution $u = -x$, so that $du = -dx$. The bounds become $u(0) = 0$ and $u(-a) = a$.

Thus, we have

$$\int_0^{-a} f(x)dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)du.$$

Then, in total, we have

$$\int_{-a}^a f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx.$$

If f is even, then $f(-u) = f(u)$, and as such

$$\int_{-a}^a f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx = \int_0^a f(u)du + \int_0^a f(u)du = 2\int_0^a f(u)du.$$

If f is odd, then $f(-u) = -f(u)$, and as such

$$\int_{-a}^a f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx = -\int_0^a f(u)du + \int_0^a f(u)du = 0.$$

□

While this won't let us directly solve an integration too often, having a bound of 0 is always super nice. But the real bonus comes from noticing that we have an odd function, and if the bounds are opposites of each other, we can just say that the whole damn thing equals 0 (as long as it's continuous).

Example 4.44. Evaluate

$$\int_{-3}^3 \cos(x) + x^6 - 10\, dx.$$

Since $f(x) = \cos(x) + x^6 - 10$ and

$$f(-x) = \cos(-x) + (-x)^6 - 10 = f(x),$$

we have an even function. As such, we can rewrite

$$\begin{aligned}\int_{-3}^{3} \cos(x) + x^6 - 10dx =& 2\int_{0}^{3} \cos(x) + x^6 - 10dx \\ =& 2 \cdot \left[\sin(x) + \frac{1}{7}x^7 - 10x\right]_0^3 \\ =& 2 \cdot \left(\sin(3) + \frac{3^7}{7} - 30\right)\end{aligned}$$

Example 4.45. Evaluate

$$\int_{-5}^{5} \frac{x^4\tan(x) - 3x}{x^2\cos(x) - \sin(x)\tan(x)}dx.$$

Omfg, EW! Well, the bounds are opposites, so let's do a symmetry check:

$$f(-x) = \frac{(-x)^4\tan(-x) - 3(-x)}{(-x)^2\cos(-x) - \sin(-x)\tan(-x)} = \frac{-x^4\tan(x) + 3x}{x^2\cos(x) - \sin(x)\tan(x)} = -f(x),$$

so we have an odd function! Thus

$$\int_{-5}^{5} \frac{x^4\tan(x) - 3x}{x^2\cos(x) - \sin(x)\tan(x)}dx = 0,$$

by our theorem. Bam!

And with that, we are done! I hope you enjoyed Calculus I.