

4.3 The Fundamental Theorem of Calculus

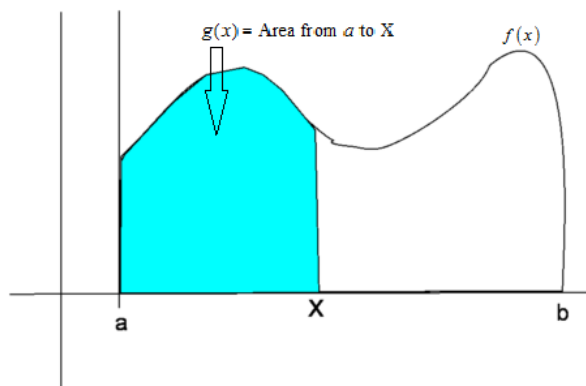
As you may have noticed, differentiation and integration are quite closely related. We finally get to that connection, the connection at the heart of calculus now – differentiation arose from the tangent line problem discussed way back when. Integration arose from the area problem previously discussed. The main theorem of this section will allow us to find areas under the curve without having to compute the limits of sums, which will be nice.

We first consider functions of the form

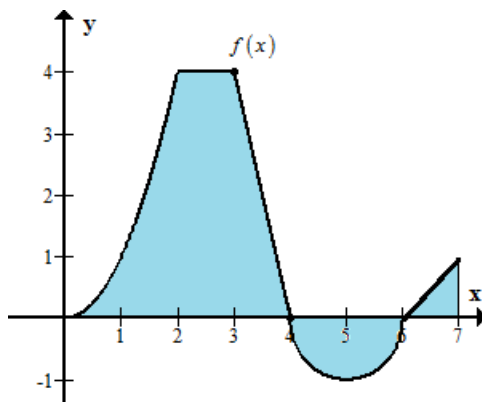
$$g(x) = \int_a^x f(t) dt,$$

where $f(t)$ is a continuous function on the interval $[a, b]$ and we also let x vary between a and b – as x is a bound on the integration, this makes sense. Note that g depends only on x , not on t . If we fix x as a number, then $\int_a^x f(t) dt$ is a number as well.

If we suppose that $f(x) \geq 0$, since $a \leq x$, $g(x)$ can be thought of as the area under the curve $f(x)$ over the interval $[a, x]$, as the “area so far.”



Example 4.17. One of the things students will struggle with is how x as a bound works and how it affects $g(x)$. Let’s take a look at the following graph.

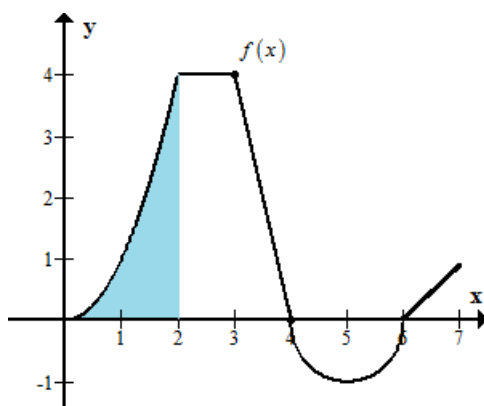


Let $g(x) = \int_0^x f(x) dx$. Find $g(0)$, $g(2)$, $g(3)$, $g(4)$, $g(5)$, $g(6)$, $g(7)$.

1. $g(0)$: This represents the area from 0 to 0 - $\int_0^0 f(x) dx = 0$. Therefore,

$$g(0) = 0$$

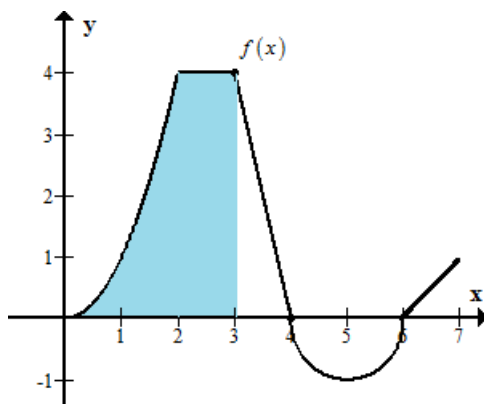
2. $g(2)$: This represents the area from 0 to 2 - $\int_0^2 f(x) dx$



We'll assume that $f(x) = x^2$ on the interval $[0, 2]$. And we know from previous sections that $\int_0^2 x^2 dx = \frac{8}{3}$. Therefore,

$$g(2) = \frac{8}{3}$$

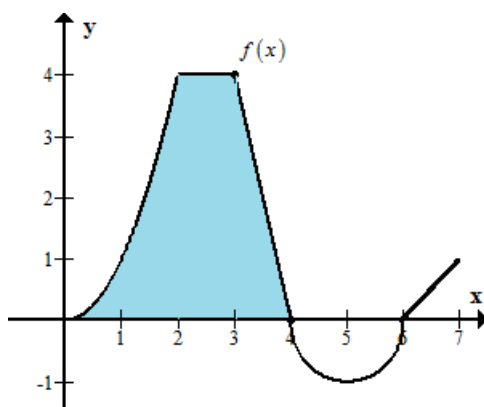
3. $g(3)$: This represents the following region:



The region from 2 to 3 is a rectangle that has an area of $1 \cdot 4 = 4$. It's a good idea to think of $g(x)$ as an accumulation of area (negative or positive) as x increases. So $g(3)$ equals all the area up to $x = 3$, which is

$$g(3) = \frac{8}{3} + 4 = \frac{20}{3} \approx 6.67$$

4. $g(4)$: This represents the following region:

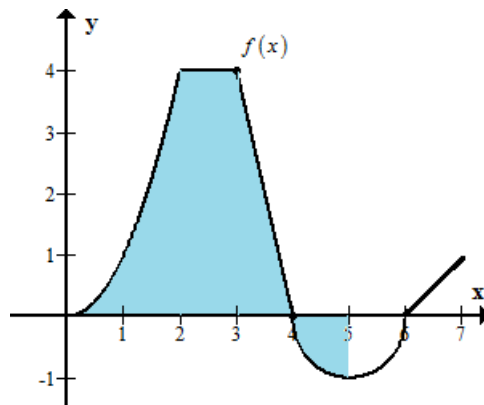


The region from 3 to 4 is a triangle with base = 1 and height = 4. Its area is $\frac{1}{2} \cdot 1 \cdot 4 = 2$. Therefore,

$$g(4) = \frac{8}{3} + 4 + 2 = \frac{26}{3} \approx 8.67$$

Notice that when we previously moved over a unit, we added an area of 4. This time we moved over a unit and added an area of 2. Even though the area function $g(x)$ increased, it didn't increase as much. In other words, (and this is very important), $g(x)$ is increasing but concave down.

5. $g(5)$: This represents the following region:

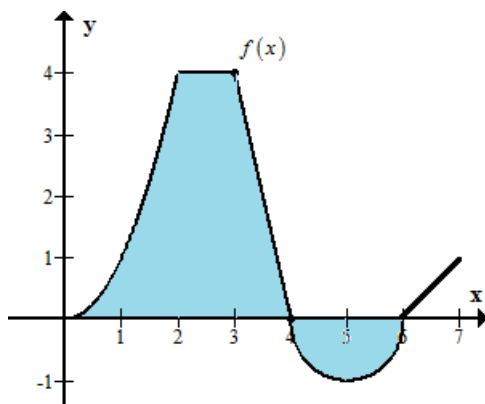


The region from 4 to 5 is a quarter of a circle with radius 1. So the area is $\frac{1}{4}\pi \approx 0.785$. But since it's below the x -axis, it's negative. Therefore,

$$g(5) = \frac{8}{3} + 4 + 2 - 0.785 = 7.88$$

Notice that $g(x)$ decreased. This should make sense since the area that we're accumulating now is negative.

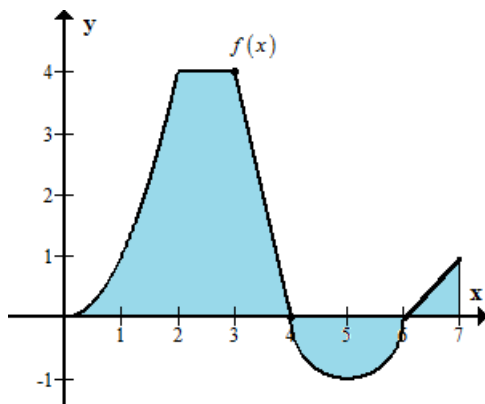
6. $g(6)$: This represents the following region:



The region from 5 to 6 is another quarter of a circle with radius 1. So the area is $\frac{1}{4}\pi \approx 0.785$. But since it's below the x -axis, it's negative. Therefore,

$$g(6) = \frac{8}{3} + 4 + 2 - 0.785 - 0.785 = 7.095$$

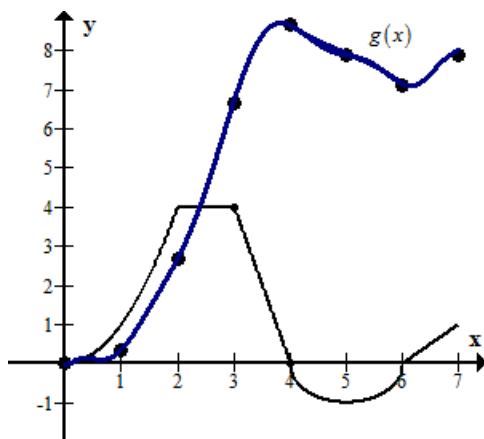
7. $g(7)$: This represents the following region:



The region from 6 to 7 is a triangle with area $\frac{1}{2}$. Therefore,

$$g(7) = \frac{8}{3} + 4 + 2 - 0.785 - 0.785 + \frac{1}{2} = 7.595$$

If you were to track $g(x)$ on a its own graph, it would look something like this



So where did $g(x)$ reach its maximum? Does it have a local minimum? From the graph it appears to have a maximum at $x = 4$. Again, if you think of $g(x)$ as the accumulation of area, $g(x)$ begins accumulating negative area after $x = 4$. Therefore $g(x)$ begins decreasing. So what did we call a point where $g(x)$ is increasing and then begins to decrease? Oh yeah... a local maximum.

It also appears to have a local minimum at $x = 6$. What's so special about $x = 6$? Notice that this is when $g(x)$ starts to accumulate positive area again. Therefore, $g(x)$ changed from decreasing to increasing and thus a local minimum.

It's actually no coincidence that the area function $g(x)$ reaches a local maximum / minimum when the original function $f(x)$ intersects the x -axis. In fact, $g(x)$ will have critical values every time $f(x) = 0$. That's odd. We know from previous chapters that $g(x)$ has

critical values when $g'(x) = 0$. Could $g'(x) = f(x)$?

Before answering this **EXTREMELY IMPORTANT** question, let's try one more example.

Example 4.18. Suppose that $f(x) = x$. Then, find

$$g(x) = \int_0^x f(t) dt = \int_0^x t dt.$$

In order to find this, we still have to treat this as a limit of a Riemann Sum: we have $\Delta x = \frac{x-0}{n} = \frac{x}{n}$ and $x_i = 0 + i\frac{x}{n} = \frac{xi}{n}$.

$$\begin{aligned} g(x) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{ix}{n} \cdot \frac{x}{n} \\ &= \lim_{n \rightarrow \infty} \frac{x^2}{n^2} \sum_{i=1}^n i \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} \left(\frac{n(n+1)}{2} \right) \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{n+1}{2n} \\ &= \frac{x^2}{2} \end{aligned}$$

Thus,

$$g(x) = \frac{x^2}{2}.$$

Note something important here:

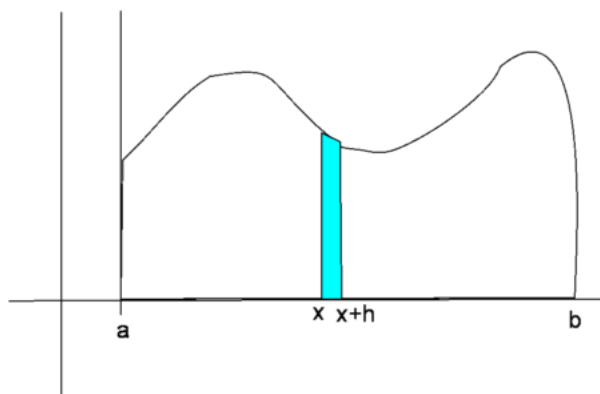
$$\frac{d}{dx} g(x) = \frac{d}{dx} \frac{x^2}{2} = \frac{2x}{2} = x = f(x).$$

In other words, $g' = f$, and if we define g to be the integral of f , then the derivative of g is f , so we view the anti-derivative AS the integral, and the interchangeability of the words make sense here. We now look to see why this may generally be true.

Begin by considering any continuous function f such that $f(x) \geq 0$. Then, define

$$g(x) = \int_a^x f(t)dt,$$

which can be interpreted as the area under the graph of f from a to x . In order to compute $g'(x)$ from the definition of a derivative, we first consider for $h > 0$, the numerator of a difference quotient, $g(x+h) - g(x)$, which is obtained by subtracting areas:



for h quite small, we have approximately a rectangle, which has height $f(x)$ and width h :

$$g(x+h) - g(x) \approx hf(x),$$

so we have

$$\frac{g(x+h) - g(x)}{h} \approx f(x).$$

Thus, if we consider the limit of the above as $h \rightarrow 0$, we have

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = f(x).$$

And thus, we have our theorem:

4.3.1 The Fundamental Theorem of Calculus, Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t) dt$$

where $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on (a, b) , and $g'(x) = f(x)$.

Proof. If we let x and $x + h$ in the interval (a, b) , then

$$\begin{aligned} g(x+h) - g(x) &= \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \\ &= \left(\int_a^x f(t) dt + \int_x^{x+h} f(t) dt \right) - \int_a^x f(t) dt \\ &= \int_x^{x+h} f(t) dt \end{aligned}$$

and thus, for $h \neq 0$, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt.$$

In order to make life a little simpler, we consider only $h > 0$ now. Since f is continuous on $[x, x+h]$, we apply the Extreme Value Theorem and have that there are numbers u and v in $[x, x+h]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values, respectively, of f on $[x, x+h]$. Then, we have

$$mh \leq \int_x^{x+h} f(t) dt \leq Mh,$$

which is to say

$$f(u)h \leq \int_x^{x+h} f(t) dt \leq f(v)h.$$

Now, since $h > 0$, we can divide by h without reversing the inequalities:

$$f(u) \leq \frac{1}{h} \int_x^{x+h} f(t) dt \leq f(v).$$

We can replace the middle of this inequality by the interior of a difference quotient, from an earlier equation:

$$f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v).$$

(We can obtain a similar inequality if $h < 0$ as well.) From here, we take a limit as $h \rightarrow 0$. Since the interval $[x, x + h]$ is squeezed down to a single point, we have $u \rightarrow x$ and $v \rightarrow x$. This gives

$$\lim_{h \rightarrow 0} f(u) = \lim_{u \rightarrow x} f(u) = f(x),$$

and

$$\lim_{h \rightarrow 0} f(v) = \lim_{v \rightarrow x} f(v) = f(x).$$

Thus, by the Squeeze Theorem, we have

$$g' = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = f(x).$$

Lastly, if $x = a$ or $x = b$, then we can view the above equation as a one-sided limit, as we would let $h \rightarrow 0^\pm$, to show that g is continuous, albeit not differentiable, on (a, b) . \square

We can rewrite part 1 of the FTC using Leibniz notation as well, and we have

$$\frac{d}{dx} \int_a^x f(t) dt = f(x),$$

for a continuous f – meaning that the derivative of an integral (with variable bounds) is the original function.

You just saw a formal proof of the FTC, I. Proofs can be very tedious and difficult to follow. It's ok if you don't follow the formal one. The explanation that lead up to stating the FTC, I is sufficient. You will see the formal proof again in an upper level advanced calculus course. Have fun with that math majors.

Example 4.19. Find

$$\frac{d}{dx} \int_1^x \sqrt{t^2 - t + 1} \cos(t) dt.$$

Since $f(t) = \sqrt{t^2 - t + 1} \cos(t)$, which is continuous for $t \geq 1$, Part 1 of the FTC gives

$$\frac{d}{dx} \int_1^x \sqrt{t^2 - t + 1} \cos(t) dt = \sqrt{x^2 - x + 1} \cos(x).$$

Example 4.20. Find

$$\frac{d}{dx} \int_x^4 e^{-t^2} dt$$

Note this isn't in the correct form. The bound x must be the upper bound. We use one of our previous integral properties to rearrange the bounds.

$$\frac{d}{dx} \left(- \int_4^x e^{-t^2} dt \right) = - \frac{d}{dx} \left(\int_4^x e^{-t^2} dt \right) = -e^{-x^2}$$

Example 4.21. Consider the function, used often in actuarial work and statistics, given by

$$S(x) = \int_0^x \sin \left(\frac{\pi t^2}{2} \right) dt.$$

By Part 1 of the FTC, we can differentiate it, so that

$$S'(x) = \sin \left(\frac{\pi x^2}{2} \right).$$

$S(x)$ is a funny function to graph – I recommend you do it sometime, if you have access to Maple or some other non-calculator software

Example 4.22. Find

$$\frac{d}{dx} \int_2^{x^3} \sin(2t) dt.$$

This is a bit more complicated, and requires more than just Part 1 of the FTC – it requires the Chain rule as well. Begin by letting $u = x^3$, and we know that $\frac{du}{dx} = 3x^2$. Then,

$$\begin{aligned} \frac{d}{dx} \int_2^{x^3} \sin(2t) dt &= \frac{d}{dx} \int_2^u \sin(2t) dt \\ &= \frac{d}{du} \left(\int_2^u \sin(2t) dt \right) \frac{du}{dx} \\ &= \sin(2u) \frac{du}{dx} \\ &= \sin(2x^3) \cdot 3x^2 \end{aligned}$$

If you're not a fan of this way, we can approach this a little differently. In order to do this same problem, what was our biggest hurdle? It was the bound x^3 . So instead, let's define

$$g(x) = \int_2^x \sin(2t) dt$$

That way, we know $g'(x) = \frac{d}{dx} \int_2^x \sin(2t) dt = \sin(2x)$.

So now consider $g(x^3)$. Note that $g(x^3) = \int_2^{x^3} \sin(2t) dt$, which is what we're trying to work with.

What's $\frac{d}{dx} [g(x^3)]$?

$$\frac{d}{dx} [g(x^3)] = g'(x^3) \cdot 3x^2 \text{ from the chain rule}$$

Since $g'(x) = \sin(2x)$, we know $g'(x^3) = \sin(2(x^3))$. Therefore,

$$\frac{d}{dx}g(x^3) = \frac{d}{dx} \int_2^{x^3} \sin(2t) dt = \sin(2x^3) \cdot 3x^2$$

Either method works. Choose one that you're comfortable with and go with it. Both are a bit weird at first, so try to work with the one that feels more natural to you.

We now come to the second part, the better part, of the FTC, which lets us avoid calculating integrals as limits of Riemann Sums, which is a long and painful process.

4.3.2 Fundamental Theorem of Calculus Part 2:

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

where F is any antiderivative of f – where F is a function such that $F' = f$.

Proof. We start by defining

$$g(x) = \int_a^x f(t)dt,$$

and from Part 1 of the FTC, we know that $g'(x) = f(x)$. Thus, g is an anti-derivative of f . If F is any other anti-derivative of f , then we had a result that said that g and F differ only by a constant, thus

$$F(x) = g(x) + c,$$

for some constant c and for $x \in (a, b)$ (which is where g is differentiable). This holds for any x on $a < x < b$. However, both F and g are continuous on $[a, b]$, and we take limits of both sides of the above equation (as $x \rightarrow a^+$ and $x \rightarrow b^-$), we also see that this is true at $x = a$ and $x = b$, so the equation holds for $x \in [a, b]$.

If we now let $x = a$ in the original formula for $g(x)$, we get

$$g(a) = \int_a^a f(t)dt = 0$$

by one of our properties of integration from earlier (same bounds). Then, we can apply the equation $F(x) = g(x) + c$ with $x = a$ and $x = b$ and we get

$$\begin{aligned} F(b) - F(a) &= (g(b) + c) - (g(a) + c) \\ &= \int_a^b f(t)dt + c - 0 - c \\ &= \int_a^b f(t)dt \end{aligned}$$

□

Thus, by Part 2 of the FTC, if we know an anti-derivative of $f(x)$, we can easily evaluate $\int_a^b f(x)dx$ simply by evaluating that anti-derivative at the bounds of integration and subtracting. It may seem a little strange that all that rectangle summing and limiting can be taken care of simply by evaluating a different function and two points, but such is the power of the integral.

To understand better why it works, suppose that $v(t)$ is the velocity of an object at time t , and $s(t)$ is the position of the object at time t , then $v(t) = s'(t)$. Thus, s is an anti-derivative of v , and as such, we can evaluate a certain pesky integral rather easily:

$$\int_a^b v(t)dt = s(b) - s(a),$$

by Part 2 of the FTC.

Example 4.23. Evaluate

$$\int_3^7 x^7 dx.$$

Remember this problem? We know that $f(x) = x^7$ is continuous everywhere, so it is definitely continuous on $[2, 7]$. So what's an anti-derivative of x^7 . Why not use the easiest one, $F(x) = \frac{1}{8}x^8$. Thus.

$$\int_3^7 x^7 dx = F(7) - F(3) = \frac{1}{8}(7)^8 - \frac{1}{8}(3)^8 = 719780.$$

Note that we could have used ANY anti-derivative, so we just went with the easiest. The same answer would have been obtained with $F(x) = \frac{1}{8}x^8 + c$ for any c , as we would've had

$$\int_3^7 x^7 dx = F(7) - F(3) = \left(\frac{1}{8}(7)^8 + c\right) - \left(\frac{1}{8}(3)^8 + c\right) = \frac{1}{8}(7)^8 - \frac{1}{8}(3)^8.$$

As a shortcut notation in the evaluation process for integration, we say that

$$F(x)|_a^b = F(b) - F(a).$$

We can write the equation in Part 2 of the FTC as

$$\int_a^b f(x) dx = F(x)|_a^b,$$

where $F' = f$.

Example 4.24. Find the area under $y = x^3 - 3$ from $x = 0$ to $x = 2$.

An antiderivative of $y = x^3 - 3$ we can use is

$$F(x) = \frac{1}{4}x^4 - 3x.$$

We find the area using Part 2 of the FTC:

$$\begin{aligned} A &= \int_0^2 x^3 - 3x \, dx = \left. \frac{1}{4}x^4 - 3x \right|_0^2 \\ &= \left(\frac{1}{4}2^4 - 3(2) \right) - \left(\frac{1}{4}0^4 - 3(0) \right) = -2 \end{aligned}$$

Example 4.25. Find the area under $y = \sin(x)$ from $x = 0$ to $x = c$, where $0 \leq c \leq \pi$.

We know that the derivative of $\cos(x) = -\sin(x)$, so to get out just $\sin(x)$, we let $F(x) = -\cos(x)$. Then

$$\frac{d}{dx} -\cos(x) = -(-\sin(x)) = \sin(x) = f(x).$$

Then

$$\int_0^c \sin(x) \, dx = -\cos(x) \Big|_0^c = -\cos(c) + \cos(0) = 1 - \cos(c).$$

Thus, really, the integral of the sine function just gives back the cosine function at c , with a bit of arithmetic involved.

Further, by letting $c = \pi$, we see that

$$\int_0^\pi \sin(x) \, dx = 1 - \cos(\pi) = 2.$$

Example 4.26. Find $\int_0^9 \sqrt{t}(1+t) \, dt$

First, we don't have an anti-derivative for a product of two functions. And you can't just anti-differentiate each and multiply them together. That's an easy way to earn 0 points. Integration is a bit more strict than differentiation. At this point, we only know the power rule,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

So we need to multiply out $\sqrt{t}(1+t)$

$$\begin{aligned} \int_0^9 \sqrt{t}(1+t) dt &= \int_0^9 \sqrt{t} + t^{3/2} dt \\ &= \int_0^9 t^{1/2} + t^{3/2} dt \\ &= \left. \frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2} \right|_0^9 \\ &= \left. \frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} \right|_0^9 \\ &= \left(\frac{2}{3}(9)^{3/2} + \frac{2}{5}(9)^{5/2} \right) - \left(\frac{2}{3}(0)^{3/2} + \frac{2}{5}(0)^{5/2} \right) \\ &= 115.2 \end{aligned}$$

Example 4.27. Find

$$\int_{-1}^3 \frac{1}{x^4} dx.$$

Well since $f(x) = x^{-4}$, we have the anti-derivative

$$F(x) = \frac{x^{-3}}{-3} = \frac{-1}{3x^3}.$$

Then, FTC Part 2 gives

$$\int_{-1}^3 \frac{1}{x^4} dx = \frac{-1}{3 \cdot 27} - \frac{-1}{-3} = \frac{-28}{81}.$$

Wait a minute, that makes no sense! We have a function $f(x) \geq 0$ for all values of x ! WTF? Well, remember the conditions of FTC Part 2 – we need the function to be continuous over $[a, b]$, and boy, oh boy do we have a discontinuity at $x = 0$, which is smack in the middle of this interval. And it's a hell of a discontinuity – it's an infinite discontinuity, one we cannot resolve in any way, shape or form. We cannot integrate the above function:

$$\int_{-1}^3 \frac{1}{x^4} dx$$

does not exist. Too bad, so sad.

Just to be clear, the anti-derivative to $f(x) = \frac{1}{x^4}$ is $-\frac{1}{3x^3}$. But we just can't evaluate it using the current endpoints.

Since the FTC has two parts, we look at them together to end this section. Recall that the FTC states that if f is continuous on $[a, b]$,

1. If $g(x) = \int_a^x f(t) dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x) dx = F(b) - F(a)$, for any anti-derivative $F(x)$ of $f(x)$.

Part 1 can be written as

$$\frac{d}{dx} \int_a^x f(t) dt = f(x),$$

which says that if we integrate and then differentiate f , we get back f . Also, since $F'(x) = f(x)$ from Part 2, we write part 2 as

$$\int_a^b F'(x) dx = F(b) - F(a).$$

Here, if we take a function, differentiate it, then integrate, we get the original function F back, but in the form of a difference between two numbers. Thus, by combining both parts of the FTC, we see that differentiation and integration are inverse functions of each other. And, armed with this little tidbit of knowledge, we will dive into integration tricks, shortcuts and the realization that life isn't as easy as it was with differentiation.