2.2 Derivative as a Function

Recall that we defined the derivative as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

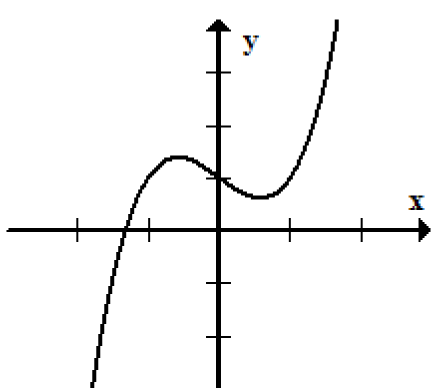
But since a is really just an arbitrary number that represents an x-value, why don't we just use x instead. We already discussed that the derivative is a function of x in the previous section.

Given $f(x)$, the derivative function is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and $f'(x)$ is interpreted as the slope any a point on $f(x)$.

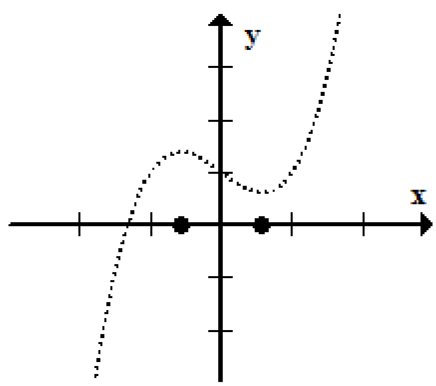
Since the derivative is a function, it also has a graph. Let's see if we can graph one. Take a look at the graph of $f(x)$ below.



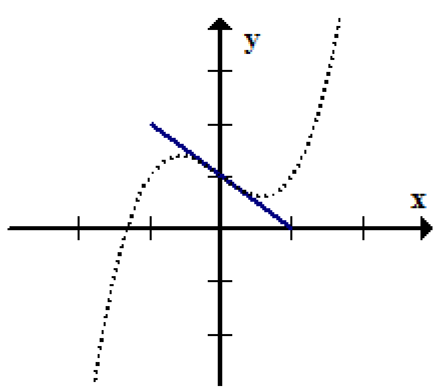
When trying to graph the derivative, say this to yourself over and over

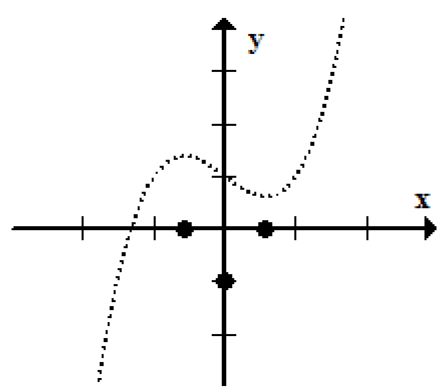
The slope on f is the y-value on f'

Start by looking for easy slopes. The easiest slope to identify is when the slope is 0. That means the graph goes horizontal (i.e,. the tangent line to the graph is horizontal). You can see this happens twice. One at $x = -0.5$ and the other at $x = 0.5$. That means the y-values on the derivative should be 0.



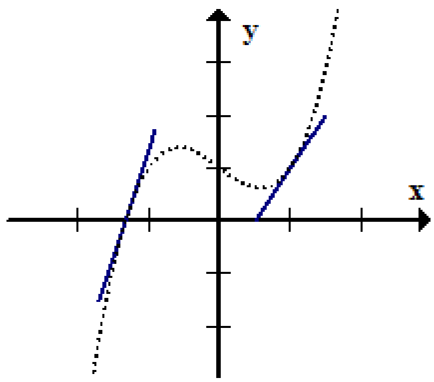
Let's try a couple other places. When $x = 0$, it looks like the graph has a slope of $m = -1$. I drew the tangent line there to get an estimate of the slope. So let's plot the point (0,-1) on $f'(x)$.



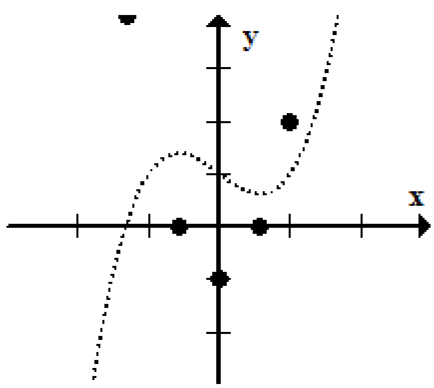


You can see we now have three points on $f'(x)$. Let's try to find two more to make sure we have a good sketch for $f'(x)$.

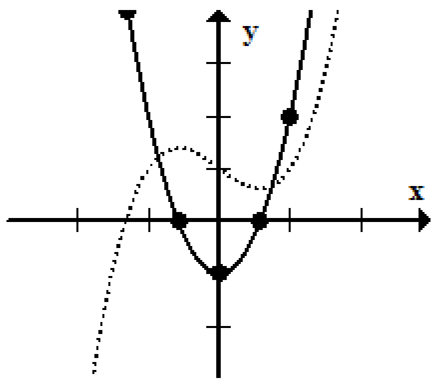
The following graph shows tangent lines at $x = -1.3$ and $x = 1$. I'd estimate the slope at $x = -1.3$ to be about 4 and the slope at $x = 1$ to be about 2. Keep in mind, we are estimating these slopes. If you want the exact slope, we'll need to find the derivative through the limit definition.



Adding the points (-1.3, 4) and (1, 2) to $f'(x)$ we have the following graph,



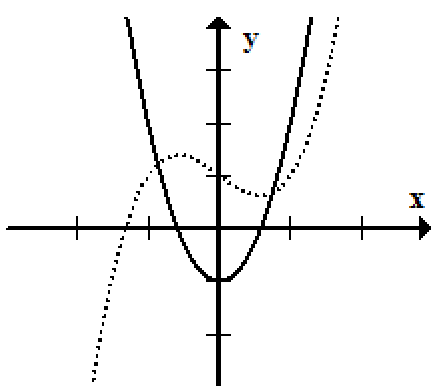
By connecting the dots, here is our graph of the derivative $f'(x)$,



So how good was our graph? Well the original graph was $f(x) = x^3 - x + 1$. Using the limit definition to find the derivative, which I leave as an exercise to you, we have

$$f'(x) = 3x^2 - 1$$

Here's what we have when we graph these two functions on the same graph,



Wow! We were pretty spot on. Now if you'd like to practice doing this, go to this site

`http://webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_try_to_graph.html`

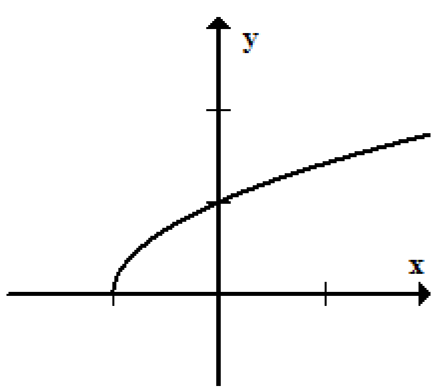
Let's do a couple more derivative problems before moving on.

Example 2.9. If $f(x) = \sqrt{x+1}$, find $f'(x)$ and its domain.

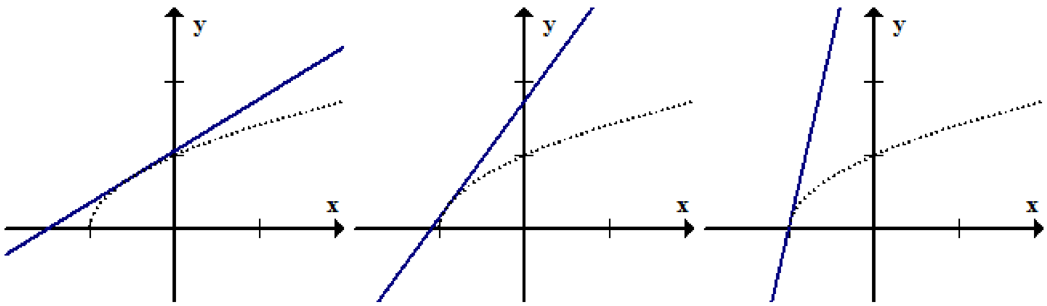
$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+1}-\sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1}+\sqrt{x+1}}{\sqrt{x+h+1}+\sqrt{x+1}} \\
&= \lim_{h\to 0} \frac{(x+h+1)-(x+1)}{h(\sqrt{x+h+1}+\sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1}+\sqrt{x+1}} \\
&= \frac{1}{2\sqrt{x+1}}
\end{aligned}$$

So $f'(x) = \dfrac{1}{\sqrt{x+1}}$, and its domain $x > -1$.

Let me note something here. The domain of $f(x) = \sqrt{x+1}$ is $x \geq -1$, but the domain of $f'(x)$ is $x > -1$. So what happened at $x = -1$? Let's take a look at the graph.



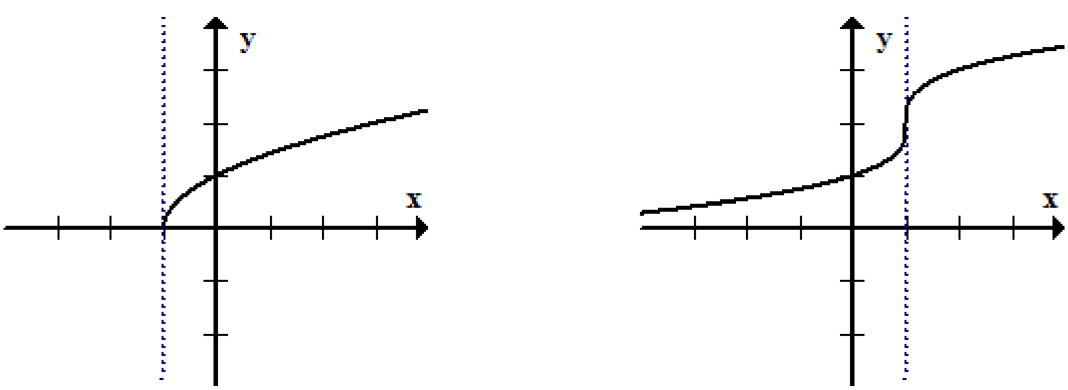
If you following the function moving right to left, you'll notice the slope getting very steep. Take a look at these tangent lines near $x = -1$.



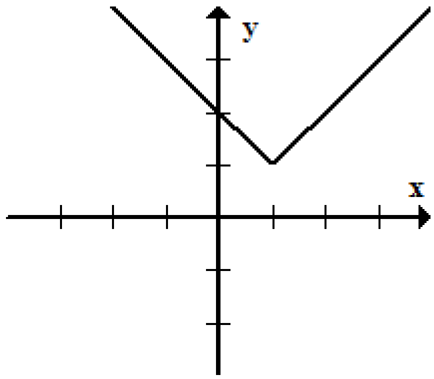
You see how the tangent lines are getting very steep as you get closer to $x = -1$. In fact, as you get closer and closer to $x = -1$, those tangent lines are heading towards a vertical line. And what's the slope of a vertical line? Anyone? Anyone? It's undefined. From algebra we always said the slope of a vertical line is not defined. That is why the derivative is not defined there. It's because that tangent line is a vertical line! And yes... I was yelling.

You might be asking yourself, "what other weird things can happen that makes a derivative not exist?" There are three main types that cause a function to not be differentiable at a point.

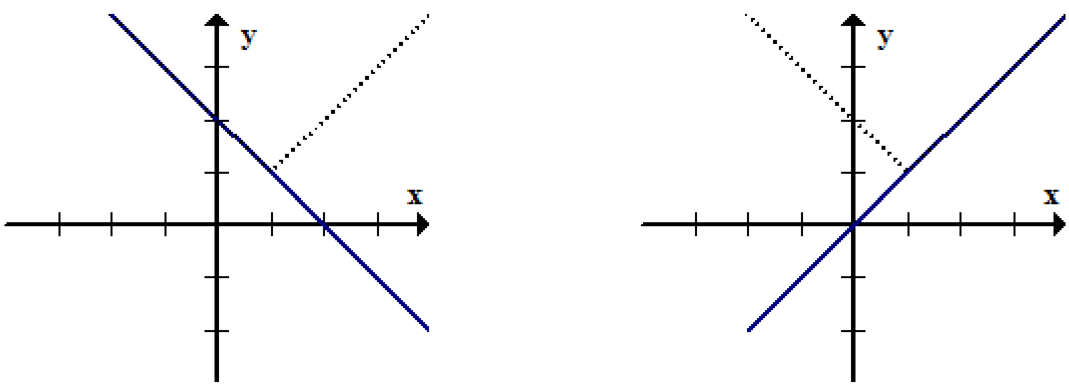
1. Vertical Tangent Lines



2. Corners



Remember that a derivative was defined to be limit of slopes. In this example, when you approach $x = 1$ from the left, what would you *expect* the slope to be? It appears to be $m = -1$. What about when you approach $x = 1$ from the right? It appears to be $m = 1$

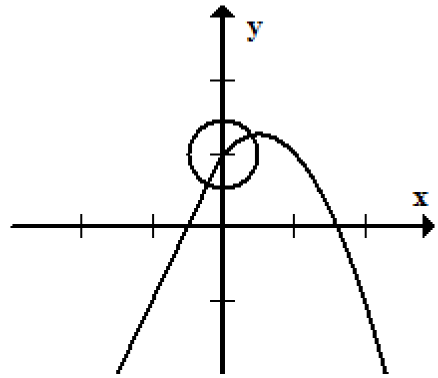


As you get closer to $x = 1$, the left and right hand slopes will not match. This is why a derivative cannot exist. The slopes must match from the left and right.

Let's take a look at another example.

Example 2.10. Let $f(x) = \begin{cases} 2x, & x < 0 \\ x - x^2, & x \geq 0 \end{cases}$

When you look at the graph you can probably guess where we have a problem with differentiability.

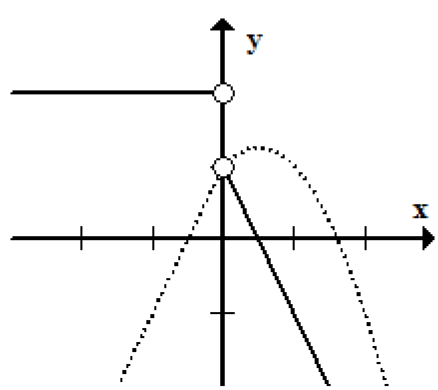


It appears we may have a corner at the point (0,1). We shouldn't be surprised. This is a piece-wise function and they almost always have problems.

Let's find $f'(x)$.

$$f'(x) = \begin{cases} 2, & x < 0 \\ 1 - 2x, & x > 0 \end{cases}$$

So if we are approaching $x = 0$ from the left, we would expect the slope to be $m = 2$. If we approach from the right, we expect the slope to be $m = 1 - 2(0) = 1$. Since these are not the same slopes, we have a corner at $x = 0$. Here's the graph of $f'(x)$.



The solid lines are $f'(x)$ whereas the dotted lines are $f(x)$. There's a more formal way of showing $f(x)$ is not differentiable at $x = 0$. But I hope the graphs help demonstrate what's going on with the slopes.

At this point, I want to introduce other notations for derivatives. Instead of $f'(x)$, you may also see,

1. y'

   This is used when you define a function like $y = x^2 - 3x + 1$

2. $\dfrac{dy}{dx}$

   Again, this is used when you define a function as $y = f(x)$.

   (a) dy means differentiate the function y

   (b) dx means differentiate with respect to x.

3. $\dfrac{df}{dx}$

   We use this notation when you define the function using $f(x)$. For example, find $\dfrac{df}{dx}$ when $f(x) = \dfrac{1}{x}$.

4. $\dfrac{d}{dx}$

   This is used when you have a function but didn't use y or $f(x)$. For example,

$$\frac{d}{dx}\left[\frac{1}{x}\right]$$

This is the same thing as saying find $f'(x)$ when $f(x) = \dfrac{1}{x}$.

5. $\dfrac{d}{dx}[f(x)]$

2.2.1 Higher Derivatives

If f is differentiable, then f' is a function.

Since f' may be a function, it could have its own derivative.

1. $f'(x) =$ first derivative

2. $(f'(x))' = f''(x) =$ second derivative

3. $(f''(x))' = f'''(x) =$ third derivative

4. $f^4(x) =$ fourth derivative

5. $f^{10}(x) =$ tenth derivative

After the third derivative we stop using $'$ (prime) notation. Writing $f''''''(x)$ seems a bit ridiculous.

We'll do some examples of these after we get our differentiation formulas.