2.8 Related Rates

The related rates section is a word problem section using implicit functions. Most of the functions in this section are functions of time t. It's probably best demonstrated through example.

Example 2.41. A stone is dropped into a pool of water. A circular ripple spreads out in the pool. The radius of the ripple increases at a rate of 5 ft / second. How fast is the area of the circle increasing? After 4 seconds?

There are two rates involved here. One is the rate the radius increases and the other is the rate the area increases. One we know and the other we don't. Our goal is to find some sort of equation or formula that relates all the information we're given.

Procedure

1. Write down all the information we have and what we need to find.

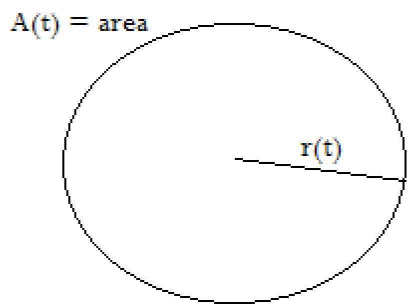
$$\text{Known: } \frac{dr}{dt} = 5 \text{ ft/s}$$
$$\text{Unknown: } \frac{dA}{dt} = ?$$

2. Find a relationship between the variables.

   Since we're dealing with circles, area, and a radius, let's use

$$A = \pi r^2$$

3. Draw a "snapshot" at some typical instant t to get an idea of what it looks like.



You can draw the picture first or after you identify some of the variables needed in the problem. It's up to you.

4. Differentiate the relationship with respect to t.

This is the tricky part. You have two variables, A and r. These variables are actually functions of time t. It might be a better idea to write

$$A(t) = \pi r(t)^2$$

$$\frac{dA}{dt} = \pi \cdot \left(2r \cdot \frac{dr}{dt}\right) = 2\pi r \frac{dr}{dt}$$

But we know $\frac{dr}{dt}$.

$$\frac{dA}{dt} = 2\pi r \cdot (5)$$

$$\frac{dA}{dt} = 10\pi r \text{ ft}^2/\text{sec}$$

We are done with the first part. Notice that the rate at which the area increases is a function of the radius (which is a function of time). So if we want to know how fast it's increasing at a specific time, we need a time.

Note: You are not allowed to plug information about specific instances until after you differentiate. For example, we are asked to find how fast the area is increasing at $t = 4$ seconds. We only use this information after we differentiated and found $\frac{dA}{dt}$.

To find how fast the area is increasing after 4 seconds, we need to know the radius after 4 seconds. Since the radius increases at a rate of 5 ft/sec, the radius should be 20 feet.

$$\frac{dA}{dt} = 10\pi(20) = 200\pi \text{ ft}^2/\text{sec}$$

Example 2.42. Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast in the radius of the balloon increasing when its diameter is 50cm.

As I mentioned in the previous example, any information about a specific point in time is not used until after you differentiate. So we do not use the information about the diameter being 50cm until later.

1. We need a formula that relates all the information together. We're dealing with a sphere and its volume. So let's use

$$V = \frac{4}{3}\pi r^3$$

2. What is known and unknown?

$$\text{Known: } \frac{dV}{dt} = 100$$

$$\text{Unknown: } \frac{dr}{dt} = ?$$

3. Differentiate $V = \frac{4}{3}\pi r^3$ with respect to t.

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2 \cdot \frac{dr}{dt}$$

4. Solve for $\frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$$

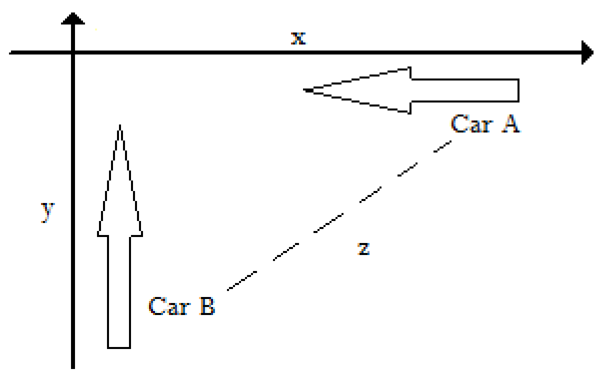
5. We want how fast the radius increases when the diameter is 50cm. We don't have a diameter in my function, but we do have the radius. If the diameter is 50 cm, the radius is 25. We also know $\frac{dV}{dt} = 100$.

$$\frac{dr}{dt} = \frac{100}{4\pi(25)^2} = 0.0127 \text{ cm/s}$$

Also note that as time goes on, $\dfrac{dr}{dt}$ is getting smaller. That means the radius keeps getting bigger, but much more slowly.

Example 2.43. Car A is traveling west at 50 mph and Car B is traveling north at 60 mph. Both are heading for the same intersection. At what rate are the cars approaching each other when Car A is .3 miles and Car B is .4 miles from the intersection?

1. Draw a picture:



Let z represent the distance between Cars A and B. The rate at which the cars approach each other is $\dfrac{dz}{dt}$.

2. What do we know?

$$\text{Known: } \frac{dx}{dt} = -50$$

$$\text{Known: } \frac{dy}{dt} = -60$$

$$\text{Unknown: } \frac{dz}{dt} = ?$$

Anyone notice that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are negative? Even though Car B is traveling up towards the intersection, velocity measures the rate of change in position. The cars are moving towards the intersection. This means their position from the intersection is decreasing, hence the negative velocity.

3. What relationship brings all of our variables together.

   Do you see from the picture we formed a right triangle. So the relationship we are going to use is

$$x^2 + y^2 = z^2$$

4. Differentiate and solve for $\frac{dz}{dt}$

$$\begin{aligned} 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} &= 2z \cdot \frac{dz}{dt} \\ x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} &= z \cdot \frac{dz}{dt} \end{aligned}$$

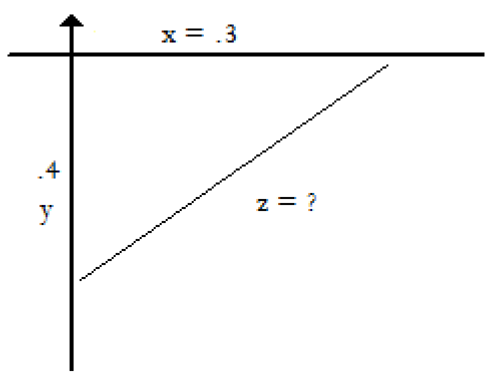
   Solving for $\frac{dz}{dt}$, we get

$$\frac{dz}{dt} = \frac{x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}}{z}$$

5. To find $\frac{dz}{dt}$ at the specific moment the question asked we need the following information,

- x
- y
- z
- $\frac{dx}{dt}$
- $\frac{dy}{dt}$

We know everything but z. So let's use that right triangle and solve for z.



$$\begin{aligned} x^2 + y^2 &= z^2 \\ (.3)^2 + (.4)^2 &= z^2 \\ .25 &= z^2 \end{aligned}$$

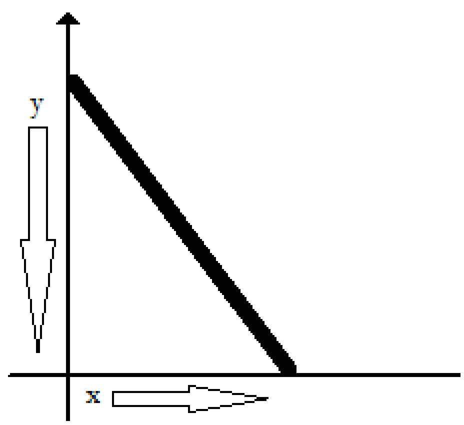
Solving for z, we get $z = .5$.

6. Plug everything in to find $\dfrac{dz}{dt}$

$$\frac{dz}{dt} = \frac{(.3)(-50) + (.4)(-60)}{.5} = -78 \text{ mph}$$

Example 2.44. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away at 1 ft/sec, how fast is the top sliding down the wall when the bottom is 6 ft from the wall?

1. Draw a picture



2. What do we know?

$$\text{Known: } \frac{dx}{dt} = 4$$
$$\text{Unknown: } \frac{dy}{dt} = ?$$

3. The relationship

$$x^2 + y^2 = 10^2$$

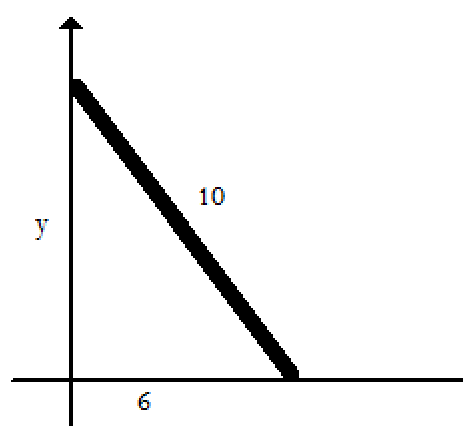
4. Differentiate with respect to t

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} \quad = \quad 0$$

Solving for $\frac{dy}{dt}$ we get

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

5. We need to find y



Using the Pythagorean Theorem, $y = 8$.

6. Plug everything in to find $\frac{dy}{dt}$

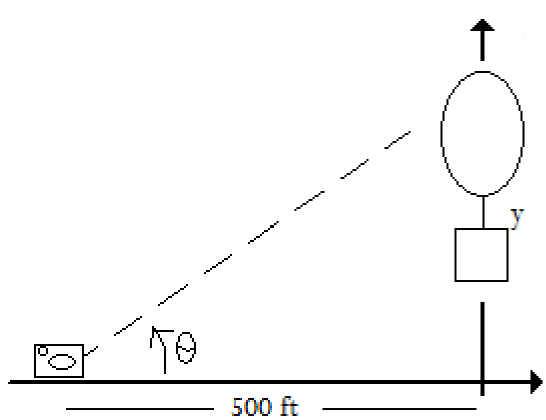
$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt} = -\frac{6}{8} \cdot 1 = -\frac{3}{4} \text{ ft/sec}$$

That's interesting. The rate at which the ladder is sliding away from the wall is not necessarily the rate at which the ladder slides down.

Also, what do you think happens when the top of the ladder gets close to the ground (i.e., when y gets close to 0)?

Example 2.45. A balloon is rising straight up from a level field and is being tracked by a camera 500 ft from the point of liftoff. At the instant when the camera's angle of elevation is $\pi/4$, that angle is increasing at a rate of 0.14 radians/min. How fast is the balloon rising at that distance?

1. Draw a picture



2. What do we know?

$$\text{Known: } \frac{d\theta}{dt} = 0.14 \text{ when } \theta = \pi/4$$
$$\text{Unknown: } \frac{dy}{dt} = ??$$

3. Relationship between the variables θ and y

$$\tan(\theta) = \frac{y}{500}$$

4. Differentiate with respect to t

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dy}{dt}$$

5. Solve for $\frac{dy}{dt}$

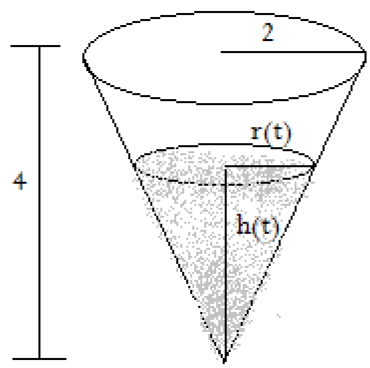
$$\frac{dy}{dt} = 500 \sec^2(\theta) \cdot \frac{d\theta}{dt}$$

6. Plug in $\theta = \pi/4$ and $\frac{d\theta}{dt} = 0.14$, and we get

$$\frac{dy}{dt} = 500 \sec^2(\pi/4) \cdot 0.14 = 140 \text{ ft/min}$$

Example 2.46. A water tank has the shape of an inverted circular cone with a base radius of 2 meter and a height of 4m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water is rising when the water is 3 m deep.

1. Let's take a look at a picture shall we?



2. What do we know?

$$\text{Known: } \frac{dV}{dt} = 2$$
$$\text{Unknown: } \frac{dh}{dt} = ??$$

3. The relationship (formula) is the volume of a circular cone

$$V = \frac{1}{3}\pi r^2 \cdot h$$

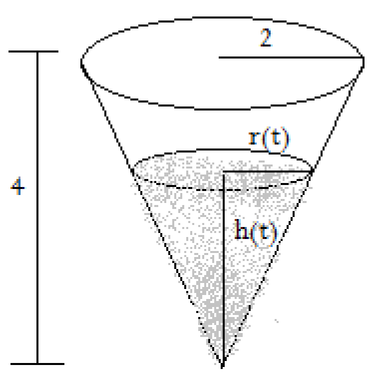
4. It may not be obvious yet but you don't want to differentiate this yet. You have three functions in there, $V(t)$, $r(t)$, and $h(t)$. After you differentiate, you'll have $\frac{dV}{dt}$, $\frac{dr}{dt}$, and $\frac{dh}{dt}$ floating around somewhere. We know $\frac{dV}{dt}$ and we're trying to solve for $\frac{dh}{dt}$, but we know nothing about $\frac{dr}{dt}$. This means we need to somehow get rid of $r(t)$. If you're still unclear, let's go ahead and differentiate what we have. You'll see what I mean.

5. Differentiate $V = \underbrace{\frac{1}{3}\pi r^2}\cdot\underbrace{h}$. BTW, it's a product of two functions, we we'll use the product rule.

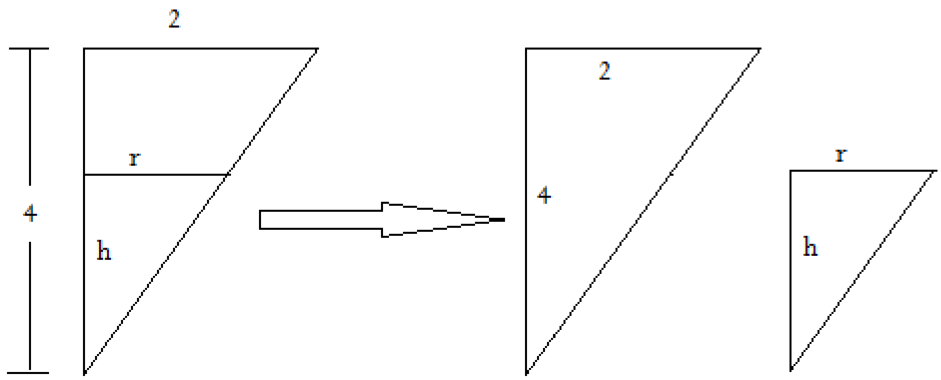
$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt} + h \cdot \left[\frac{2}{3}\pi r \cdot \frac{dr}{dt}\right]$$

Do you see what I mean about $\frac{dr}{dt}$? We have no information on it. So again, we need to get rid of $r(t)$.

6. If you look at the picture again,



you'll notice we actually have two similar triangles.



Using this, we can write $\dfrac{r}{h} = \dfrac{2}{4}$. Solving for r and we get

$$r = \frac{h}{2}$$

7. Rewrite our volume formula.

$$V = \frac{1}{3}\pi r^2 \cdot h = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 \cdot h = \frac{1}{12}\pi h^3$$

8. Now we differentiate,

$$\frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

9. Now plug in $\dfrac{dV}{dt} = 2$, $h = 3$, and solve for $\dfrac{dh}{dt}$.

$$2 = \frac{\pi}{4} \cdot 3^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8}{9\pi} = .28 \text{ m/min}$$

When the height of the water is 3 m high, the rate at which the height is rising is 0.28 m/min.