1.9 The Precise Definition of a Limit

This section introduces us to a very formal way of defining a limit. It's really hit or miss on whether students catch on. If you are a math major, you will encounter this topic again. There are two types of problems in this section. For now, I'm only going to cover one of them. You can read the textbook on the 2nd type.

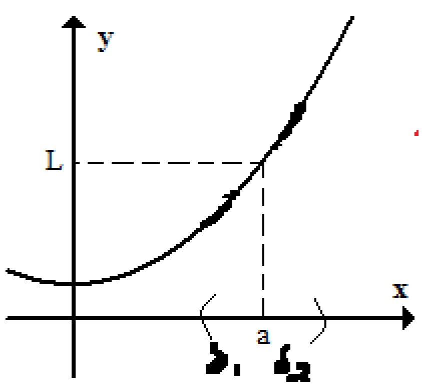
First, we need to introduce some new notation.

- ϵ - epsilon
- δ - delta
- $|f(x) - L| \Rightarrow$ represents the distance between $f(x)$ and L.
- $|x - a| \Rightarrow$ represents the distance between x and a.

Definition 1.4 (Precise Defn of a Limit)**.** Let f be a function on some open interval that contains a. We say the limit of $f(x)$ as $x \to a$ is L

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$



Let's break down the definition and apply it to the graph.

1. ϵ and δ represent distances. These numbers are very small positive numbers.

2. $|x - a| < \delta$ means x gets really close to a (close enough so the distance between x and a is less than δ).

3. What our definition really states is,

   If x is really close to a, inside a tiny interval around a, then the y-values associated with those x-values must be within ϵ from L. That's the meaning of $|f(x) - L| < \epsilon$.

   If you are claiming that the limit is L, then I can pick any distance from L and you should be able to show me that there are x-values whose y-values are that close to L.

   I'm finding it a bit difficult to explain this in typed notes. My explanation will be better in class since I can point to things and answer questions. Let's just move on to an example.

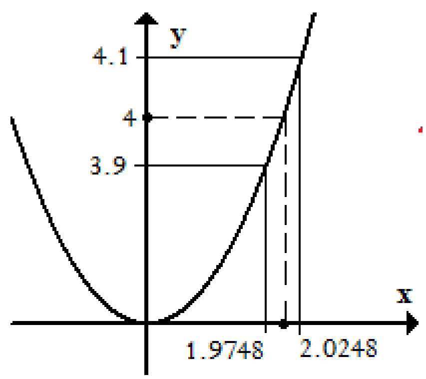
NOTE TO ME: Show them a graph where the limit exists and a graph where a limit does not exist.

Example 1.30. Consider $\lim_{x \to 2} x^2 = 4$

Let $\epsilon = 0.1$

So if I give you ϵ (the distance away from the limit value) which creates an interval 3.9 and 4.1, you must have me an interval around $x = 2$ that gets me y-values between 3.9

and 4.1. Based on the graph,you can see I found the x-values that give me a y-value of 3.9 and 4.1. This means if I choose any x-value between 1.9748 and 2.0248, I will get a y-value between 3.9 and 4.1.



δ is how close the x-values need to be from 2 to guarantee y-values between 3.9 and 4.1.

Let $\delta_1 = |1.9784 - 2| = 0.0252$ and $\delta_2 = |2.0248 - 2| = 0.0248$. The formal δ is the minimum of $\{\delta_1, \delta_2\}$.

So $\delta = 0.0248$.

Summary: If I'm $\epsilon = .1$ away from $L = 4$, then I only need to be $\delta = 0.0248$ away from $x = 2$.

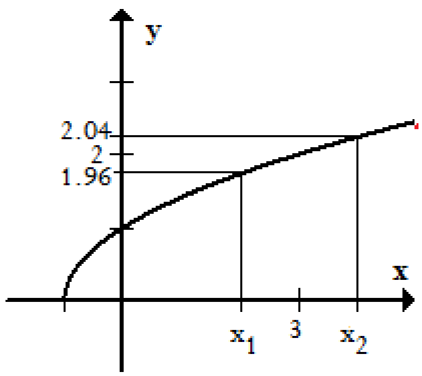
Every ϵ can have a different δ. As ϵ gets smaller, so does δ. You can imagine that by just making the lines at 3.9 and 4.1 get closer together. But this is the point. Given any ϵ, if you can find me a δ that can guarantee me y-values close to L, then L is truly the limit value.

I understand that I'm not explaining this that great in my notes. If you have any questions, please come see me. It so much easier to explain this in person.

Example: Consider $\lim_{x\to 3} \sqrt{x+1}$.

Given $\epsilon = 0.04$, find δ. What does this mean?

We are allowed to be at a distance of $\epsilon = 0.04$ away from the limit value, which is 2. The question is, "How far can x be from 3 so that the y-values are less than 0.04 away from $L = 2$." Let's take a look at the graph.



So what's x_1 and x_2?

You can use your calculator or solve the following equations:

$$\sqrt{x_1+1} = 1.96$$

and

$$\sqrt{x_2+1} = 2.04$$

Solving these two equations, we have $x_1 = 2.8416$ and $x_2 = 3.1616$.

Now δ_1 represents the distance from x_1 to 3 and δ_2 represents the distance from x_2 to 3.

$$\delta_1 = |2.8416 - 3| = 0.1584$$

$$\delta_2 = |3.1616 - 3| = 0.1616$$

And $\delta = min\{\delta_1, \delta_2\}$

So,

$$\delta = 0.1584$$