3.2 Mean Value Theorem

Let's begin the section with an easier extension of the Mean Value Theorem.

Theorem 3.3 (Rolle's Theorem). *1. f is continuous on the interval $[a, b]$.*

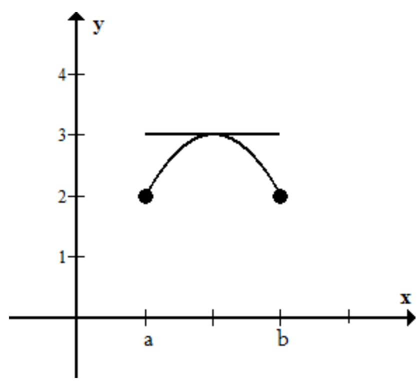
2. f is differentiable on the interval (a, b).

3. $f(a) = f(b)$

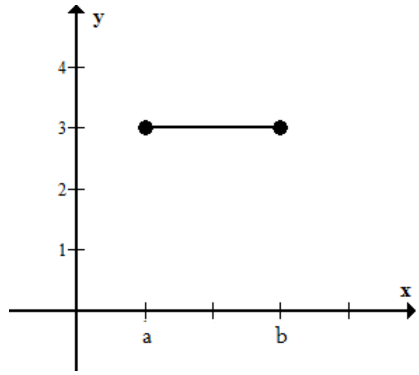
If the above conditions are satisfied, then there exists a c in (a, b) such that $f'(c) = 0$ (f must have a slope of 0).

It's easy to see that this is true by looking at some examples.

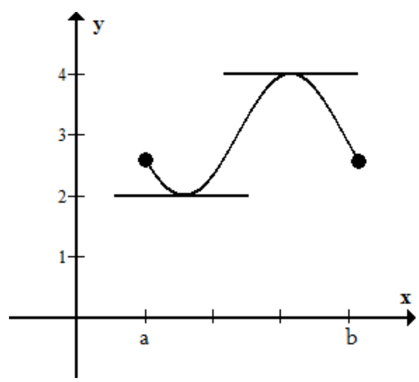
Example 3.4.



Notice that all conditions to Rolle's Theorem are satisfied. We can clearly see an x-value in (a, b) where $f'(x) = 0$.



This example is a bit trivial since the entire line has a slope of 0. But, it satisfied the conditions so there must be a place where $f'(x) = 0$. It just so happens we have a slope of 0 everywhere.



There really isn't a limit to how many places we could have a slope of 0. The theorem doesn't guarantee the number of places. It only guarantees the existence of at least one.

So what are some of the uses for Rolle's Theorem? First let's recap what's really happening. If $f(a) = f(b)$, then we must have a slope of 0 somewhere between a and b. Another way of saying this is $f(x)$ must turn around somewhere between a and b.

Example 3.5. Show that $x^3 + x - 1 = 0$ has exactly one real root (solution).

Solution:

Well to verify this, we first need to show it has at least one root. A while ago, we discussed the Intermediate Value Theorem. One of the reasons we used this theorem was to show when a function has a root.

By the way, $f(x) = x^3 + x - 1$ satisfies the conditions of the theorem (it's continuous). Now let $x = 0$ and $x = 2$. If you're not given an interval, make one up.

$$x = 0: \qquad f(0) = 0^3 + 0 - 1 = -1 \text{ (negative value)}$$

$$x = 2: \qquad f(2) = 2^3 + 2 - 1 = 9 \text{ (positive value)}$$

Since $f(0)$ and $f(2)$ have opposite signs, IVT says we must have a root between 0 and 2.

Now to have a second root, our function must turn around. Assume $f(x)$ has two roots. In order to have two roots (i.e., reach the x-axis again), it must turn around.

A differentiable function can only turn around when the slope is 0. So let's check out the derivative.

$$f'(x) = 3x^2 + 1 = 0$$

If we try to set this function equal to 0

$$3x^2 + 1 = 0$$

we see it has no solution. So what does that mean for us?

If $f'(x) \neq 0$, then the function has no way of turning around. If the function cannot turn around, it has no way of getting a second root.

We showed that $f(x)$ has at least one root by using the Intermediate Value Theorem. Now we showed using Rolle's Theorem that $f(x)$ can't turn around, thus eliminating the possibility for a second root. Therefore, $x^3 + x - 1$ has only one root.

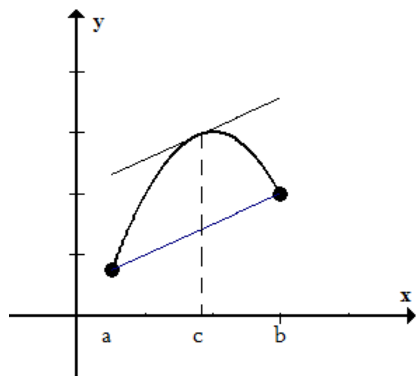
Theorem 3.4 (The Mean Value Theorem)**.** .

1. *f is continuous on $[a, b]$.*

2. *f is differentiable on (a, b).*

Then there exists a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

The picture below shows why this makes sense. Note that $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the line connecting the endpoints.



This theorem states that there must be another place in (a, b) that has a tangent line with the same slope as the secant line connection $(a, f(a))$ to $(b, f(b))$.

Example 3.6. Let $f(x) = x^3 + 3x^2$ on the interval $[-5, 1]$. Since f is a polynomial, it is continuous and differentiable on the domain.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(-5)}{1 - (-5)} = 9$$

The Mean Value Theorem tells me there exists a number c somewhere between -5 and 1 where $f'(c) = 9$. Let's go ahead and find out.

$$f'(c) = 3c^2 + 6c = 9$$

Solving $3c^2 + 6c = 9$, we get

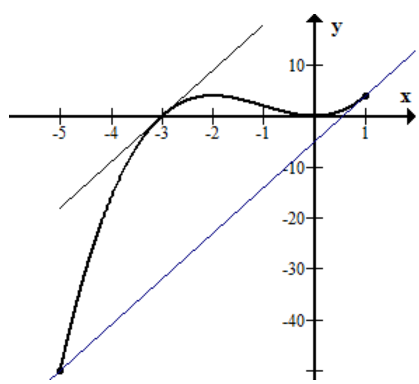
$$3c^2 + 6c - 9 = 0$$

$$3(c^2 + 2c - 3) = 0$$

$$(c+3)(c-1) = 0$$

The solutions are $c = -3$ and $c = 1$. Unfortunately, $c = 1$ is not between -5 and 1, so we don't count it. So according to the mean value theorem, the only value in $(-5, 1)$ that gives us a slope of 9 is at $x = -3$.

Let's take a look at the graph and verify this.



Example 3.7. Show $x^3 - 15x + c = 0$ has at most one root in $[-2, 2]$.

1. Note that it doesn't ask us if there are any roots. It just wants us to show there can't be two or more.

2. Let's assume our equation does have two roots in $[-2, 2]$. That means there are two points, let's call them x_1 and x_2, such that $f(x_1) = f(x_2) = 0$.

3. Rolle's Theorem says there must exist a c in (x_1, x_2) such that $f'(c) = 0$. Let's find $f'(x)$.

$$f'(x) = 3x^2 - 15$$

4. Solving for x, we have $x = \pm\sqrt{5}$. Both of these x values are NOT in $[-2, 2]$.

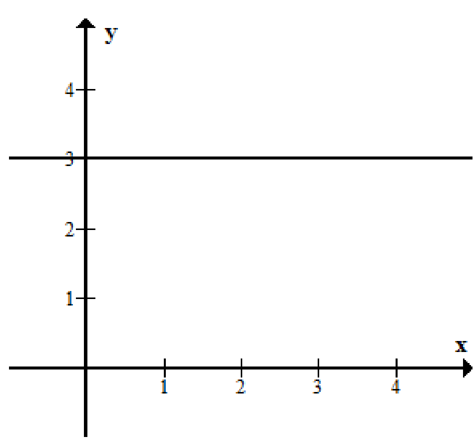
5. We conclude that it must be impossible for our equation to have two roots in $[-2, 2]$. And with that, we are done.

6. Just to be clear, all we've done is shown the equation cannot have two or more roots. We don't know (nor care) if it has zero or one root.

Here are some other theorems that will help us in the next few sections.

Theorem 3.5. *If $f'(x) = 0$ for all x in (a, b), then f is constant.*

Example 3.8. Consider the following function $y = f(x) = 3$ (a constant function).



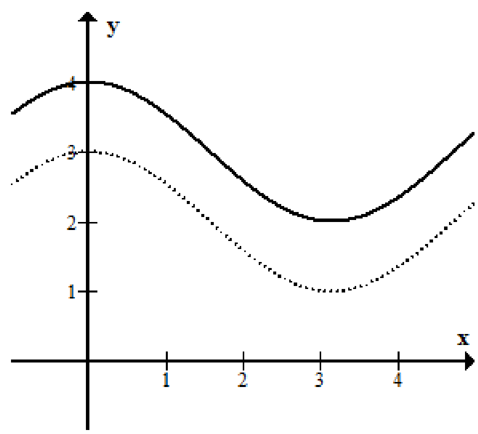
This theorem should make sense. If the function wasn't constant (a horizontal line), how could $f'(x) = 0$?

Let's move onto another theorem. This theorem will show its face again when we get into integration. Don't worry, that's still a ways away.

Theorem 3.6. *If $f'(x) = g'(x)$ for all x in (a, b), then $f(x) - g(x)$ is a constant. Another way of concluding the theorem is,*

$$f(x) = g(x) + C$$

Again, let's take a look at another graph to check this out. Note that if two functions have the exact same derivative (i.e., they have the exact same slopes), they have to look the same. They just might be off by a constant.



Note, the difference between $f(x)$ and $g(x)$ is the same.