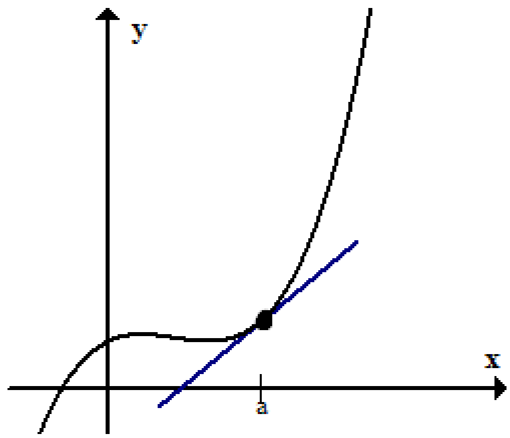
2.9 Linear Approximations and Differentials

2.9.1 Linear Approximation

Consider the following graph,



Recall that this is the tangent line at $x = a$. We had the following definition,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

So for x close to a, we have the following

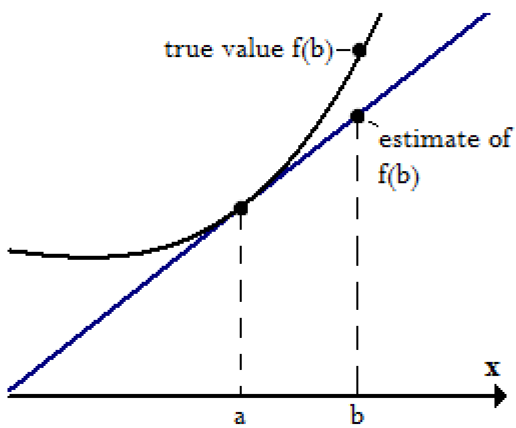
$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

After rearranging the terms, we get an estimate for $f(x)$ when x is near a.

$$f(x) = f(a) + f'(a)(x - a)$$

This is called the linearization of $f(x)$ near $x = a$ or linear approximation of $f(x)$ near $x = a$. You may not recognize it, but this is the equation of the tangent line at $x = a$. It's just written with different notation.

So how can this be useful? Suppose you wanted to find the value $f(b)$ where b is really close to a. Instead of using the function $f(x)$ to evaluate it, we can just the tangent line. From the graph, you can see that the tangent line and the function $f(x)$ look very similar if you focus only on the area near $x = a$. Here's that graph focused near $x = a$.



You can see the true value of $f(b)$ and the estimate you get from the tangent line are pretty close. Now as you move away from $x = a$, the tangent line and the function deviate quite a bit. So a linear approximation is only useful when evaluating near $x = a$.

Example 2.47. Find the linearization of $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Recall the linearization of $f(x)$ near $x = a$ is $f(x) \approx L(x) = f(a) + f'(a)(x-a)$. So what do we need?

- a
- $f(a)$
- $f'(x)$
- $f'(a)$

We know $a = 1$, and $f(a) = f(1) = \sqrt{1+3} = 2$, and

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

so,

$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

Putting all this together, we get

$$f(x) \approx L(x) = 2 + \frac{1}{4}(x-1)$$

So how do we estimate $\sqrt{4.05}$? If you plug in $a = 1$, we get $\sqrt{4}$. So what do you have to plug in for x to get $\sqrt{4.05}$? You need to plug in $x = 1.05$, right?

$$\sqrt{4.05} = f(1.05) \approx L(1.05) = 2 + \frac{1}{4}(1.05 - 1)$$

$$\sqrt{4.05} \approx 2.0125$$

Now, let's do the same thing and estimate $\sqrt{3.98}$. Again, we need to figure out what is x. What do we need to plug in to $\sqrt{x+3}$ to get $\sqrt{3.98}$? I'm hoping you say $x = 0.98$. If you did, that's great.

$$\sqrt{3.98} = f(3.98) \approx L(3.98) = 2 + \frac{1}{4}(.98 - 1) = 1.995$$

So how close are our estimates? Let's find their relative error.

$$\text{Relative Error: } \left|\frac{\text{TRUE } - \text{ APPROX}}{\text{APPROX}}\right|$$

So for $\sqrt{3.98}$, the relative error is

$$\text{RE } = \left|\frac{\sqrt{3.98} - 1.995}{\sqrt{3.98}}\right| = 0.000003141$$

And for $\sqrt{4.05}$, the relative error is

$$\text{RE} = \left|\frac{\sqrt{4.05} - 2.0125}{\sqrt{4.05}}\right| = 0.00001929$$

These aren't bad estimates. Now what if I try to estimate $\sqrt{10}$ using the current linearization formula?

$$\sqrt{10} = f(7) \approx L(7) = 2 + \frac{1}{4}(7-1) = 3.5$$

The relative error is

$$\text{RE} = \left|\frac{\sqrt{10} - 3.5}{\sqrt{10}}\right| = .106797 \text{ or } 10.6797\%$$

Obviously, this estimate isn't as good as the previous two. However, it's still not bad. If you want to experiment more, try estimating $\sqrt{100}$ or something higher.

Example 2.48. Using linearization, estimate $\sin(\pi/180)$.

1. First, the whole point of learning about linearization is to estimate something complicated with something easy. In the last example, we used $\sqrt{4}$ to estimate $\sqrt{4.05}$.

2. Let $f(x) = \sin(x)$.

3. We need a value for a. $\pi/180$ is close to 0, so let's use $a = 0$. As a bonus (more of a requirement really), we know $\sin(0)$.

4. Recall that $L(x) = f(a) + f'(a)(x - a)$. We need to find $f'(a)$.

$$f'(x) = \cos(x)$$

so,

$$f'(0) = \cos(0) = 1$$

5. Let's put this all together to find $L(x)$.

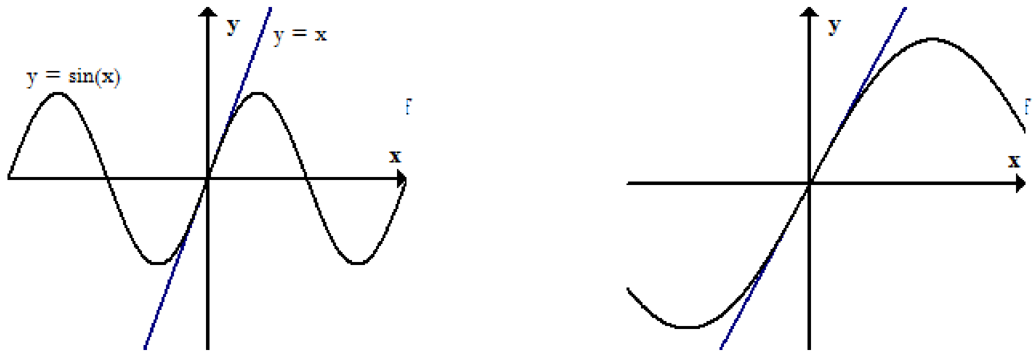
$$\sin(x) \approx L(x) = \sin(0) + \cos(0)(x - 0) = x$$

That's interesting. When x is near 0, we just showed $\sin(x) \approx x$.

Our objective was to estimate $\sin(\pi/180)$. Based on what we just saw,

$$\sin(\pi/180) \approx \pi/180$$

I'd like to verify $\sin(x) \approx x$ by looking at the graph.



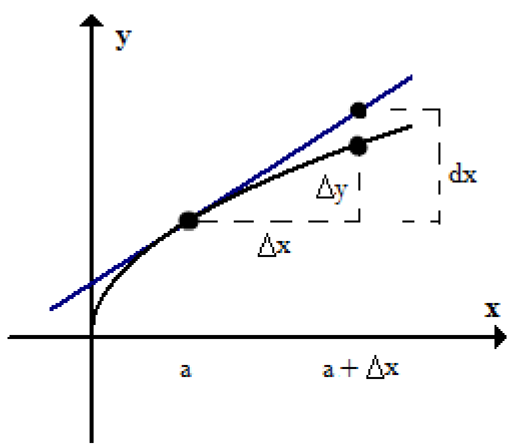
The graph on the right is zoomed in near $x = 0$ to show you that the function $f(x) = x$ is a good approximation for $f(x) = \sin(x)$.

2.9.2 Differentials

Differentials can be used to do exactly what we just did with linearization. Differentials help us estimate the change in function values. Let's look at some new notation.

- Δx - is the true change in x
- dx - is our independent variable that represents the chagen in x. We let $dx = \delta x$.
- Δy - is the true change in y
- dy - is the estimated change in y

Without looking at the graph yet, does it make sense that the change in y depends on the change in x? This is why dx is an independent variable and dy is the dependent variable.



Remember that Δy is the true change in y. Based on the graph, we see that $\Delta y = f(x + \Delta x) - f(x)$. This involves us knowing the exact value of $f(x + \Delta x)$. Back in the linearization part, knowing $f(x + \Delta x)$ is like knowing $\sqrt{4.05}$. Instead I want to estimate Δy with dy. Let's start with something we know,

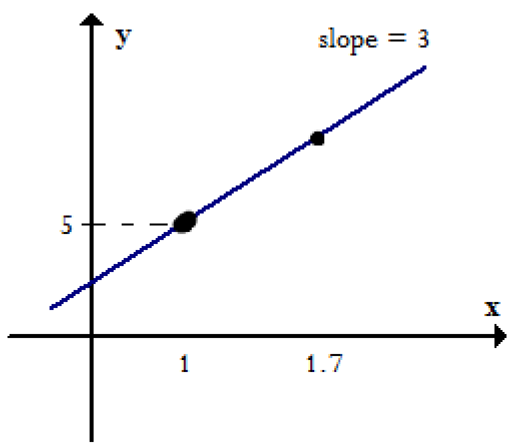
$$\frac{dy}{dx} = f'(x)$$

This was another way of notating the derivative. Now if we treat dx as an independent variable, we can rearrange this as

$$dy = f'(x) \cdot dx$$

But this should make sense. $\frac{dy}{dx}$ is the derivative at x. It represents the slope (i.e, the change in y for every unit change in x). Recall that dx is the change in x.

Let's look at a simple example. Suppose I know at $x = 1$, the y-value is 5. How can I use the differential formula to estimate $f(1.7)$?



If the slope is 3 and the change in x, dx, is 0.7, then the change in y from $x = 1$ to $x = 1.7$ is $3 \cdot 0.7 = 2.1$.

$$dy = f'(1) \cdot dx = 3 \cdot 0.7 = 2.1$$

So if you're starting at a y-value of 5 and move up 2.1 units, then the new y-value is 7.1.

Example 2.49. Let's go back to a previous problem and estimate $\sqrt{4.05}$ using the function $f(x) = \sqrt{x+3}$.

1. We need a starting point. We will use $a = 1$ since $f(1) = \sqrt{4}$ and that's close to what we want.

2. Next, we need to find dx. What do we have to plug into $f(x)$ to get $\sqrt{4.05}$? We need to plug in $x = 1.05$. However, dx is just the change in x from your starting point a. So $dx = 0.05$.

3. Now we need the slope, $f'(1)$, at $x = 1$.

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

So,

$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

4. So the estimated change in y is

$$dy = f'(1) \cdot (0.05) = \frac{1}{4} \cdot 0.05 = .0125$$

Horray! We have the estimated change in y. But what is this really? dy tells us how much y changes from the original y value. For us, the original y value is $\sqrt{4} = 2$. So

$$\sqrt{4.05} \approx \sqrt{4} + dy = 2 + 0.0125 = 2.0125$$

Guess what? That's exactly what we got using linearization!