1.8 Limit Laws and Methods

Theorem 1.1. *If $f(x)$ is a rational function (polynomials are rational BTW) and a is in the domain of f, then*

$$\lim_{x \to a} f(x) = f(a)$$

So what does this mean?

It basically means, if you can plug in $x = a$ and get a 'legal' number, then that's the limit value.

Example 1.20. Find $\displaystyle\lim_{x \to 3} \frac{x}{x+2}$

Since $f(x)$ is a rational function, $\displaystyle\lim_{x \to 3} \frac{x}{x+2} = \frac{3}{3+2} = \frac{3}{5}$

Example 1.21. $\displaystyle\lim_{x \to 5} \sqrt[3]{\frac{4x+44}{6x-29}}$

This one isn't a rational function, but the idea is still the same. If it's a nice, non-piecewise function and a is in the domain, then just plug in $x = 5$. If you get a nice value out, then that's the limit.

$$\begin{aligned} \lim_{x \to 5} \sqrt[3]{\frac{4x+44}{6x-29}} &= \sqrt[3]{\frac{4(5)+44}{6(5)-29}} \\ &= \sqrt[3]{64} \\ &= 4 \end{aligned}$$

1.8.1 The Limit Laws

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

1. $\lim_{x\to a} f \pm g = L \pm M$

   The limit of a sum or difference of functions is the sum or difference of their limit values.

2. $\lim_{x\to a} c \cdot f(x) = c \cdot L = c \cdot \lim_{x\to a} f(x)$

   The constant can be pulled out of a limit.

3. $\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = L \cdot M$

   Please note that you can only break a limit of a product into a product of limits provided the individual limits both exist.

4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, provided $\lim_{x\to a} g(x) \neq 0$.

5. $\lim_{x\to a} c = c$

   This should make sense. We are taking a limit of a constant. No matter what x is, the function value is always c.

1.8.2 Limits by Cancellation

Example 1.22. Consider the limit $\lim_{x\to 1} \dfrac{x^2 - 6x + 5}{x - 1}$

If you get $\frac{0}{0}$, it means the limit *may* still exist.

What do you do? Factor, simplify, and try plugging in $x = 1$ again.

1. Factor:

$$\lim_{x\to 1} \frac{x^2 - 6x + 5}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x-5)}{x-1}$$

2. Simplify:

$$= \lim_{x\to 1} x - 5$$

3. Try again:

$$\begin{aligned} &= 1 - 5 \\ &= -4 \end{aligned}$$

Example 1.23. Find $\lim_{x\to -3} \frac{2x^2 + 5x - 3}{x^2 - 9}$

1. Plug in $x = -3$

$$\lim_{x\to -3} \frac{2x^2 + 5x - 3}{x^2 - 9} = \frac{0}{0}$$

2. Since we get $\frac{0}{0}$, we factor $f(x)$

$$\lim_{x\to -3} \frac{2x^2+5x-3}{x^2-9} = \lim_{x\to -3} \frac{(2x-1)(x+3}{(x-3)(x+3)}$$

3. Simplify

$$\lim_{x\to -3} \frac{(2x-1)(x+3}{(x-3)(x+3)} = \lim_{x\to -3} \frac{2x-1}{x-3}$$

4. Try again. Plug in $x = -3$

$$\lim_{x\to -3} \frac{2x-3}{x-3} = \frac{2(-3)-1}{-3-3} = \frac{7}{6}$$

5. Final Answer:

$$\lim_{x\to -3} \frac{2x^2+5x-3}{x^2-9} = \frac{7}{6}$$

Example 1.24. Find $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$

1. Plug in $x = 1$

$$\lim_{x\to 1}\frac{x^2-1}{x^2-2x+1}=\frac{0}{0}$$

2. Since we get $\dfrac{0}{0}$, we factor $f(x)$

$$\lim_{x\to 1}\frac{x^2-1}{x^2-2x+1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-1)}$$

3. Simplify

$$\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-1)}=\lim_{x\to 1}\frac{x+1}{x-1}$$

4. Try again. Plug in $x=1$

$$\lim_{x\to 1}\frac{x+1}{x-1}=\frac{1+1}{1-1}\to\frac{2}{0}$$

5. We get a $\dfrac{\text{non-zero}}{0}$. This means the limit is $-\infty$, ∞, or DNE. Since it's a general limit, let's find the left and right hand limits.

6. Left Hand Limit

$$\lim_{x \to 1^-} \frac{x+1}{x-1} = -\infty$$

7. Right Hand Limit

$$\lim_{x \to 1^+} \frac{x+1}{x-1} = \infty$$

8. Since

$$\lim_{x \to 1^+} \frac{x+1}{x-1} \neq \lim_{x \to 1^-} \frac{x+1}{x-1}$$

$$\lim_{x \to 1} \frac{x^2-1}{x^2-2x+1} \text{ does not exist}$$

1.8.3 Limits by Simplifying

Example 1.25. Evaluate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$

1. Plug in $h = 0$.

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \frac{0}{0}$$

Recall, $\frac{0}{0}$ means something weird is happening. There isn't anything to factor, but we can foil and simplify.

2. Foil the top and simplify

$$\begin{aligned}
\lim_{h\to 0} \frac{(3+h)^2-9}{h} &= \lim_{h\to 0} \frac{(9+6h+h^2)-9}{h} \\
&= \lim_{h\to 0} \frac{h(6+h)}{h} \\
&= \lim_{h\to 0} 6+h
\end{aligned}$$

3. Try again. Plug in $h = 0$

$$\lim_{h\to 0} 6+h = 6+0 = 6$$

Therefore,

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = 6$$

1.8.4 Limits using the conjugate

Example 1.26. Find $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$

1. Plug in $x = 0$.

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \frac{0}{0}$$

Since we get $\frac{0}{0}$, we must try to simplify this somehow.

2. Multiply by the conjugate

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} &= \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} \\ &= \lim_{x\to 0} \frac{(4+x)+2\sqrt{4+x}-2\sqrt{4+x}-4}{x\left(\sqrt{4+x}+2\right)}\end{aligned}$$

3. Simplify

$$\begin{aligned} &= \lim_{x\to 0} \frac{(4+x)+2\sqrt{4+x}-2\sqrt{4+x}-4}{x\left(\sqrt{4+x}+2\right)} \\ &= \lim_{x\to 0} \frac{4+x-4}{x\left(\sqrt{4+x}+2\right)} \\ &= \lim_{x\to 0} \frac{x}{x\left(\sqrt{4+x}+2\right)} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2}\end{aligned}$$

4. Try again. Plug in $x = 0$.

$$\lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Therefore, $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x} = \dfrac{1}{4}$

1.8.5 Limits using common denominators

Example 1.27. Find $\lim\limits_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right)$

1. As always, try plugging in $t = 0$.

$$\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) = \frac{1}{0} - \frac{1}{0}$$

   These are not numbers you can subtract. We are dividing by zero, so we should try to simplify this. Let's simplify by making this into one fraction.

2. Make into one fraction by using a common denominator

$$\begin{aligned}
\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right) &= \lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t(t+1)}\right) \\
&= \lim_{t\to 0} \left(\frac{1}{t} \cdot \frac{t+1}{t+1} - \frac{1}{t(t+1)}\right) \\
&= \lim_{t\to 0} \left(\frac{t+1}{t(t+1)} - \frac{1}{t(t+1)}\right) \\
&= \lim_{t\to 0} \left(\frac{t}{t(t+1)}\right) \\
\text{Simplify} &= \lim_{t\to 0} \frac{1}{t+1} \\
\text{Plug in } t = 0 &= \frac{1}{0+1} \\
&= 1
\end{aligned}$$

Therefore, $\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=1$

Example 1.28. Find $\lim_{x\to a}\frac{\frac{5x}{x+3}-\frac{5a}{a+3}}{x-a}$

1. As usual, plug in $x=a$.

$$\lim_{x\to a}\frac{\frac{5x}{x+3}-\frac{5a}{a+3}}{x-a}=\frac{\frac{5a}{a+3}-\frac{5a}{a+3}}{a-a}=\frac{0}{0}$$

2. Since we get $\frac{0}{0}$, let's simplify by multiplying the top and bottom by a common denominator.

$$\begin{aligned}\lim_{x\to a}\frac{\frac{5x}{x+3}-\frac{5a}{a+3}}{x-a} &= \lim_{x\to a}\frac{\frac{5x}{x+3}-\frac{5a}{a+3}}{x-a}\cdot\frac{(a+3)(x+3)}{(a+3)(x+3)}\\ &= \lim_{x\to a}\frac{5x(a+3)-5a(x+3)}{(x-a)(a+3)(x+3)}\\ &= \lim_{x\to a}\frac{5ax+15x-5ax-15a}{(x-a)(a+3)(x+3)}\\ &= \lim_{x\to a}\frac{15x-15a}{(x-a)(a+3)(x+3)}\end{aligned}$$

3. At this point I hope you see that when you have $\frac{0}{0}$, the goal is to find a way to divide out (cancel) the denominator that's giving us 0. Look back at all the other problems we've done and you'll see that's exactly what we did to evaluate the limits. This problem is no different. We need a way to divide out or cancel the denominator $x-a$.

If our goal is to cancel the $x-a$ on the bottom, does the numerator have a factor of $x-a$? It sure does.

$$\begin{aligned}\lim_{x\to a}\frac{15x-15a}{(x-a)(a+3)(x+3)} &= \lim_{x\to a}\frac{15(x-a)}{(x-a)(a+3)(x+3)}\\ &= \lim_{x\to a}\frac{15}{(a+3)(x+3)}\end{aligned}$$

4. Try again. Plug in $x=a$.

$$\lim_{x\to a}\frac{15}{(a+3)(x+3)} = \frac{15}{(a+3)^2}$$

If you want a really hard one, evaluate the following:

$$\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)$$

This one uses common denominators, conjugates, and simplifying. Show that

$$\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right)=\frac{-1}{2}$$

Theorem 1.2. *If $f(x)\le g(x)$ when x is near a then*

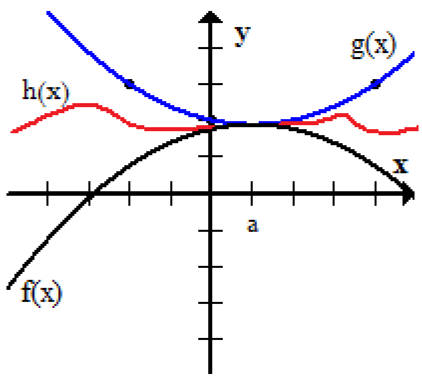
$$\lim_{x\to a}f(x)\le\lim_{x\to a}g(x)$$

1.8.6 Squeeze Theorem

If $f(x) \leq h(x) \leq g(x)$ when x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} h(x) = L$$

A picture would be good here to show you how the squeeze theorem works.

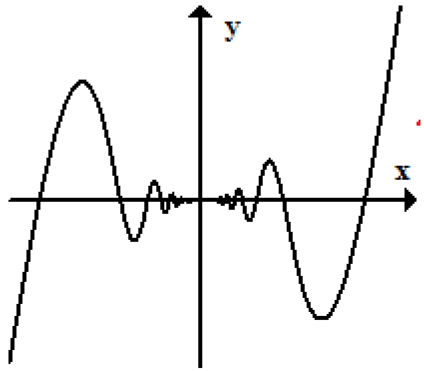


If $h(x)$ must remain between $f(x)$ and $g(x)$ when x approaches a, and $f(x)$ and $g(x)$ are expected to be the same value as x approaches a, then it forces $h(x)$ to approach the same limit value. In other words, where else can $h(x)$ go if it must stay between f and g near $x = a$?

Based on the graph, you can probably guess why we call it the **Squeeze Theorem**.

Example 1.29. Show $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Let's take a look at the graph.



Based on the graph, we would probably guess $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

We need to find two functions, one bigger than $x^2 \sin(1/x)$ and one less than $x^2 \sin(1/x)$.

We can't just plug in $x = 0$ because we'll divide by 0. Since $\sin(1/x)$ is what's giving us trouble, let's try to get rid of it. Let's bound $\sin(1/x)$.

We know from earlier that

$$-1 \leq \sin(u) \leq 1$$

which means

$$-1 \leq \sin(1/x) \leq 1$$

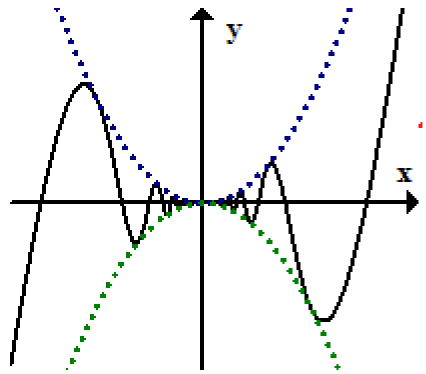
Next, we multiply all sides by x^2.

$$-x^2 \leq x^2 \sin(1/x) \leq x^2$$

So how does this help us?

1. Notice the middle function is our function from the limit. Think of this as our $h(x)$.

2. Notice that $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$. Think of these as our $f(x)$ and $g(x)$, where $f(x) \leq g(x)$ near $x = 0$. Let's graph all three functions.



Can you see how $-x^2$ and x^2 trap or 'squeeze' $x^2 \sin(1/x)$?

3. The inequality shows us that $f(x) \leq h(x) \leq g(x)$ near $x = 0$ and $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$. Using the Squeeze Theorem, we can also conclude $\lim_{x \to 0} x^2 \sin(1/x) = 0$